CSPs: Arc Consistency

CPSC 322 – CSPs 3

Textbook §4.5
Lecture Overview

1. Recap
2. Arc Consistency
CSPs as Search Problems

We map CSPs into search problems:

- **nodes**: assignments of values to a subset of the variables
- **neighbours** of a node: nodes in which values are assigned to one additional variable
- **start node**: the empty assignment (no variables assigned values)
- **goal node**: a node which assigns a value to each variable, and satisfies all of the constraints

Note: the path to a goal node is not important
An example of solving a CSP using depth-first search, with pruning whenever a partial assignment violates a constraint.
Constraint Networks

- A constraint network:
  - Two kinds of nodes in the graph
    - one node for every variable
    - one node for every constraint
  - Edges run between variable nodes and constraint nodes: they indicate that a given variable is involved in a given constraint
Lecture Overview

1 Recap

2 Arc Consistency
Recap Arc Consistency

Arc Consistency

**Definition**

An arc \( \langle X, r(X, \bar{Y}) \rangle \) is arc consistent if for each value of \( X \) in \( \text{dom}(X) \) there is some value \( \bar{Y} \) in \( \text{dom}(\bar{Y}) \) such that \( r(X, \bar{Y}) \) is satisfied.

- In symbols, \( \forall X \in \text{dom}(X), \exists \bar{Y} \in \text{dom}(\bar{Y}) \) such that \( r(X, \bar{Y}) \) is satisfied.
- A network is arc consistent if all its arcs are arc consistent.
- If an arc \( \langle X, \bar{Y} \rangle \) is not arc consistent, all values of \( X \) in \( \text{dom}(X) \) for which there is no corresponding value in \( \text{dom}(\bar{Y}) \) may be deleted from \( \text{dom}(X) \) to make the arc \( \langle X, \bar{Y} \rangle \) consistent.
  - This removal can never rule out any models (do you see why?)
Arc Consistency Algorithm

- Consider the arcs in turn making each arc consistent.
  - Arcs may need to be revisited whenever the domains of other variables are reduced.

- Regardless of the order in which arcs are considered, we will terminate with the same result: an arc consistent network.

- Worst-case complexity of this procedure:
  - let the max size of a variable domain be $d$
  - let the number of constraints be $e$
  - complexity is $O(ed^3)$

- Some special cases are faster
  - e.g., if the constraint graph is a tree, arc consistency is $O(ed)$
Arc Consistency Outcomes

- Three possible outcomes (when all arcs are arc consistent):
  - One domain is empty ⇒ no solution
  - Each domain has a single value ⇒ unique solution
  - Some domains have more than one value ⇒ may or may not be a solution
    - in this case, arc consistency isn’t enough to solve the problem: we need to perform search
Arc Consistency Algorithm

procedure AC(V, dom, R)

Inputs
V: a set of variables
dom: a function such that dom(X) is the domain of variable X
R: set of relations to be satisfied

Output
arc consistent domains for each variable

Local
D_X is a set of values for each variable X

for each variable X do
D_X ← dom(X)
end for each

TDA ← \{(X, r) | r ∈ R is a constraint that involves X\}

while TDA ≠ {} do
select \langle X, r \rangle ∈ TDA;
TDA ← TDA − \{(X, r)\};
ND_X ← \{x | x ∈ D_X and there is \( \bar{y} \) ∈ D_Y such that \( r(x, \bar{y}) \)\};
if ND_X ≠ D_X then
TDA ← TDA \cup \{(Z, r') | r' ≠ r and r' involves X and Z ≠ X\};
D_X ← ND_X;
end if
end while
return \{D_X : X is a variable\}
end procedure
Revisiting Edges

- When we change the domain of a variable $X$ in the course of making an arc $\langle X, r \rangle$ arc consistent, we add every arc $\langle z, r' \rangle$ where $r'$ involves $X$ and:
  - $r \neq r'$
  - $z \neq X$

- Thus we don’t add back the same arc:
  - This makes sense—it’s definitely arc consistent.
Revisiting Edges

- When we change the domain of a variable $X$ in the course of making an arc $\langle X, r \rangle$ arc consistent, we add every arc $\langle Z, r' \rangle$ where $r'$ involves $X$ and:
  - $r \neq r'$
  - $Z \neq X$

- We don’t add back other arcs involving the same variable $X$
  - We’ve just *reduced* the domain of $X$
  - If an arc $\langle X, r \rangle$ was arc consistent before, it will still be arc consistent
    - in the “for all” we’ll just check fewer values
Revisiting Edges

When we change the domain of a variable $X$ in the course of making an arc $\langle X, r \rangle$ arc consistent, we add every arc $\langle Z, r' \rangle$ where $r'$ involves $X$ and:

- $r \neq r'$
- $Z \neq X$

We don’t add back other arcs involving the same constraint and a different variable:

- Imagine that such an arc—involving variable $Y$—had been arc consistent before, but was no longer arc consistent after $X$’s domain was reduced.
- This means that some value in $Y$’s domain could satisfy $r$ only when $X$ took one of the dropped values.
- But we dropped these values precisely because there were no values of $Y$ that allowed $r$ to be satisfied when $X$ takes these values—contradiction!