Logic: Soundness and Completeness of Bottom-Up Proofs

CPSC 322 – Logic 4

Textbook §5.2
Lecture Overview

1. Recap

2. Soundness of Bottom-Up Proofs

3. Completeness of Bottom-Up Proofs
A proof is a mechanically derivable demonstration that a formula logically follows from a knowledge base.

Given a proof procedure, $KB \vdash g$ means $g$ can be derived from knowledge base $KB$.

Recall $KB \models g$ means $g$ is true in all models of $KB$.

**Definition (soundness)**

A proof procedure is **sound** if $KB \vdash g$ implies $KB \models g$.

**Definition (completeness)**

A proof procedure is **complete** if $KB \models g$ implies $KB \vdash g$. 
Bottom-up Ground Proof Procedure

One rule of derivation, a generalized form of *modus ponens*:

*If “$h \leftarrow b_1 \land \ldots \land b_m$” is a clause in the knowledge base, and each $b_i$ has been derived, then $h$ can be derived.*

You are **forward chaining** on this clause.

(This rule also covers the case when $m = 0$.)
Recap Soundness of Bottom-Up Proofs
Completeness of Bottom-Up Proofs

Bottom-up proof procedure

\[ KB \vdash g \text{ if } g \subseteq C \] at the end of this procedure:

\[
C := \{\};
\]

**repeat**

- **select** clause “\( h \leftarrow b_1 \land \ldots \land b_m \)” in \( KB \) such that
  - \( b_i \in C \) for all \( i \), and \( h \notin C \);

\[
C := C \cup \{h\}
\]

**until** no more clauses can be selected.
Example

\[
\begin{align*}
a & \leftarrow b \land c. \\
a & \leftarrow e \land f. \\
b & \leftarrow f \land k. \\
c & \leftarrow e. \\
d & \leftarrow k. \\
e. \\
f & \leftarrow j \land e. \\
f & \leftarrow c. \\
j & \leftarrow c.
\end{align*}
\]
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Soundness of bottom-up proof procedure

If \( KB \vdash g \) then \( KB \models g \).

- Suppose there is a \( g \) such that \( KB \vdash g \) and \( KB \not\models g \).
- Let \( h \) be the first atom added to \( C \) that’s not true in every model of \( KB \).
- Suppose \( h \) isn’t true in model \( I \) of \( KB \).
- There must be a clause in \( KB \) of form

\[
h \leftarrow b_1 \land \ldots \land b_m
\]

Each \( b_i \) is true in \( I \). \( h \) is false in \( I \). So this clause is false in \( I \).
- Therefore \( I \) isn’t a model of \( KB \). Contradiction: thus no such \( g \) exists.
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Minimal Model

We can use proof procedure to find a model of $KB$.

- First, observe that the $C$ generated at the end of the bottom-up algorithm is a fixed point.
  - further applications of our rule of derivation will not change $C$. 

  **Definition (minimal model)**

  Let the minimal model $I$ be the interpretation in which every element of the fixed point $C$ is true and every other atom is false.

  **Claim:**

  $I$ is a model of $KB$. 

  **Proof:**

  Assume that $I$ is not a model of $KB$. Then there must exist some clause $h ← b_1 ∧ \ldots ∧ b_m$ in $KB$ (having zero or more $b_i$'s) which is false in $I$.

  This can only occur when $h$ is false and each $b_i$ is true in $I$.

  If each $b_i$ belonged to $C$, we would have added $h$ to $C$ as well.

  Since $C$ is a fixed point, no such $I$ can exist.
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- This can only occur when $h$ is false and each $b_i$ is true in $I$.
- If each $b_i$ belonged to $C$, we would have added $h$ to $C$ as well.
- Since $C$ is a fixed point, no such $I$ can exist.
Completeness

If $KB \models g$ then $KB \vdash g$.

- Suppose $KB \models g$. Then $g$ is true in all models of $KB$.
- Thus $g$ is true in the minimal model.
- Thus $g$ is generated by the bottom up algorithm.
- Thus $KB \vdash g$. 