Reasoning Under Uncertainty: Marginal and Conditional Independence

CPSC 322 Lecture 25

March 21, 2007
Textbook §9.2 – §9.3
Lecture Overview

1 Recap

2 Marginal Independence

3 Conditional Independence
Conditioning

- Probabilistic conditioning specifies how to revise beliefs based on new information.
- You build a probabilistic model taking all background information into account. This gives the prior probability.
- All other information must be conditioned on.
- If evidence $e$ is all of the information obtained subsequently, the conditional probability $P(h|e)$ of $h$ given $e$ is the posterior probability of $h$. 

Conditional Probability

The conditional probability of formula $h$ given evidence $e$ is

$$P(h|e) = \frac{P(h \land e)}{P(e)}$$

Chain rule:

$$P(f_1 \land f_2 \land \ldots \land f_n) = \prod_{i=1}^{n} P(f_i|f_1 \land \ldots \land f_{i-1})$$

Bayes’ theorem:

$$P(h|e) = \frac{P(e|h) \times P(h)}{P(e)}.$$
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2 Marginal Independence

3 Conditional Independence
Definition (marginal independence)

Random variable \( X \) is marginally independent of random variable \( Y \) if, for all \( x_i \in \text{dom}(X) \), \( y_j \in \text{dom}(Y) \) and \( y_k \in \text{dom}(Y) \),

\[
P(X = x_i | Y = y_j) = P(X = x_i | Y = y_k) = P(X = x_i).
\]

That is, knowledge of \( Y \)'s value doesn’t affect your belief in the value of \( X \).
Examples of marginal independence

- The probability that the Canucks will win the Stanley Cup is independent of whether light $l_1$ is lit.
  - remember the diagnostic assistant domain—the picture will recur in a minute!
- Whether there is someone in a room is independent of whether a light $l_2$ is lit.
- Whether light $l_1$ is lit is not independent of the position of switch $s_2$. 
Conditional Independence

- Sometimes, two random variables might not be marginally independent. However, they can become independent after we observe some third variable.

**Definition**

Random variable $X$ is conditionally independent of random variable $Y$ given random variable $Z$ if, for all $x_i \in \text{dom}(X)$, $y_j \in \text{dom}(Y)$, $y_k \in \text{dom}(Y)$ and $z_m \in \text{dom}(Z)$,

$$P(X = x_i | Y = y_j \land Z = z_m) = P(X = x_i | Y = y_k \land Z = z_m) = P(X = x_i | Z = z_m).$$

- That is, knowledge of $Y$’s value doesn’t affect your belief in the value of $X$, given a value of $Z$. 

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Conditional Independence Example

- Kevin separately phones two students, Alice and Bob.
- To each, he tells the same number, \( n_k \in \{1, \ldots, 10\} \).
- Due to the noise in the phone, Alice and Bob each imperfectly (and independently) draw a conclusion about what number Kevin said.
- Let the numbers Alice and Bob think they heard be \( n_a \) and \( n_b \) respectively.
- Are \( n_a \) and \( n_b \) marginally independent?
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Due to the noise in the phone, Alice and Bob each imperfectly (and independently) draw a conclusion about what number Kevin said.
Let the numbers Alice and Bob think they heard be $n_a$ and $n_b$ respectively.
Are $n_a$ and $n_b$ marginally independent?
- No: we’d expect (e.g.) $P(n_a = 1|n_b = 1) > P(n_a = 1)$. 

Why are $n_a$ and $n_b$ conditionally independent given $n_k$?
Because if we know the number that Kevin actually said, the two variables are no longer correlated.

\[ P(n_a = 1|n_b = 1, n_k) = P(n_a = 1|n_k) \]
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Why are $n_a$ and $n_b$ conditionally independent given $n_k$?
- Because if we know the number that Kevin actually said, the two variables are no longer correlated.
- e.g., $P(n_a = 1|n_b = 1, n_k = 2) = P(n_a = 1|n_k = 2)$
Example domain (diagnostic assistant)
More examples of conditional independence

- Whether light $l_1$ is lit is independent of the position of light switch $s_2$ given whether there is power in wire $w_0$.
  - two random variables that are not marginally independent can still be conditionally independent

- Every other variable may be independent of whether light $l_1$ is lit given whether there is power in wire $w_0$ and the status of light $l_1$ (if it’s ok, or if not, how it’s broken).
More examples of conditional independence

- The probability that the Canucks will win the Stanley Cup is independent of whether light $l_1$ is lit given whether there is outside power.
  - sometimes, when two random variables are marginally independent, they’re also conditionally independent given a third variable.

- But not always...
  - Let $C_1$ be the proposition that coin 1 is heads; let $C_2$ be the proposition that coin 2 is heads; let $B$ be the proposition that coin 1 and coin 2 are both either heads or tails.
  - $P(C_1|C_2) = P(C_1)$: $C_1$ and $C_2$ are marginally independent.
  - But $P(C_1|C_2, B) \neq P(C_1|B)$: if I know both $C_2$ and $B$, I know $C_1$ exactly, but if I only know $B$ I know nothing.
  - Hence $C_1$ and $C_2$ are not conditionally independent given $B$. 