Reasoning Under Uncertainty: Hidden Markov Models

CPSC 322 Lecture 29

March 22, 2006
Textbook §9.5

Lecture Overview

Recap

Variable Elimination Example

Hidden Markov Models

Probability of a conjunction

- What we know: the factors $P(X_i|pX_i)$.
- Using the chain rule and the definition of a belief network, we can write $P(X_1, \ldots, X_n)$ as $\prod_{i=1}^{n} P(X_i|pX_i)$. Thus:

$$
\begin{aligned}
&P(Z, Y_1 = v_1, \ldots, Y_j = v_j) \\
&\quad = \sum_{Z_k} \cdots \sum_{Z_1} P(X_1, \ldots, X_n)_{Y_1 = v_1, \ldots, Y_j = v_j}. \\
&\quad = \sum_{Z_k} \cdots \sum_{Z_1} \prod_{i=1}^{n} P(X_i|pX_i)_{Y_1 = v_1, \ldots, Y_j = v_j}.
\end{aligned}
$$

Summing out a variable efficiently

To sum out a variable Z_j from a product $f_1, \ldots, f_k$ of factors:

- Partition the factors into
  - those that don't contain Z_j, say $f_1, \ldots, f_i$,
  - those that contain Z_j, say $f_{i+1}, \ldots, f_k$

We know:

$$\sum_{Z_j} f_1 \times \cdots \times f_k = (f_1 \times \cdots \times f_i) \left(\sum_{Z_j} f_{i+1} \times \cdots \times f_k \right).$$

- $\left(\sum_{Z_j} f_{i+1} \times \cdots \times f_k \right)$ is a new factor; let's call it f'.
- Now we have:

$$\sum_{Z_j} f_1 \times \cdots \times f_k = f_1 \times \cdots \times f_i \times f'.$$

- Store f' explicitly, and discard $f_{i+1}, \ldots, f_k$. Now we've summed out Z_j.

Variable elimination algorithm

To compute $P(Z|Y_1 = v_1 \wedge \ldots \wedge Y_j = v_j)$:

- Construct a factor for each conditional probability.
- Set the observed variables to their observed values.
- For each of the other variables $Z_i \in \{Z_1, \ldots, Z_k\}$, sum out Z_i
- Multiply the remaining factors.
- Normalize by dividing the resulting factor $f(Z)$ by $\sum_Z f(Z)$.

Lecture Overview

Recap

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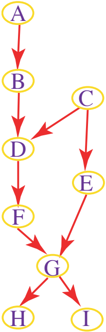
Hidden Markov Models

Variable elimination example

Compute $P(G|H = h_1)$. Elimination order: A, C, E, H, I, B, D, F

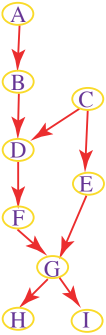
- $P(G, H) = \sum_{A,B,C,D,E,F,I} P(A, B, C, D, E, F, G, H, I)$
- $P(G, H) = \sum_{A,B,C,D,E,F,I} P(A) \cdot P(B|A) \cdot P(C) \cdot P(D|B, C) \cdot P(E|C) \cdot P(F|D) \cdot P(G|F, E) \cdot P(H|G) \cdot P(I|G)$



Variable elimination example

Compute $P(G|H=h_1)$. Elimination order: A, C, E, H, I, B, D, F

- $P(G,H) = \sum_{A,B,C,D,E,F,I} P(A) \cdot P(B|A) \cdot P(C) \cdot P(D|B,C) \cdot P(E|C) \cdot P(F|D) \cdot P(G|F,E) \cdot P(H|G) \cdot P(I|G)$
- Eliminate A: $P(G,H) = \sum_{B,C,D,E,F,I} f_1(B) \cdot P(C) \cdot P(D|B,C) \cdot P(E|C) \cdot P(F|D) \cdot P(G|F,E) \cdot P(H|G) \cdot P(I|G)$



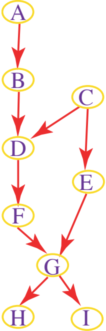
- $f_1(B) := \sum_{a \in dom(A)} P(A=a) \cdot P(B|A=a)$

Variable elimination example

Compute $P(G|H=h_1)$. Elimination order: A, C, E, H, I, B, D, F

- $P(G,H) = \sum_{B,C,D,E,F,I} f_1(B) \cdot P(C) \cdot P(D|B,C) \cdot P(E|C) \cdot P(F|D) \cdot P(G|F,E) \cdot P(H|G) \cdot P(I|G)$
- Eliminate C: $P(G,H) =$

  $\sum_{B,D,E,F,I} f_1(B) \cdot f_2(B,D,E) \cdot P(F|D) \cdot P(G|F,E) \cdot P(H|G) \cdot P(I|G)$

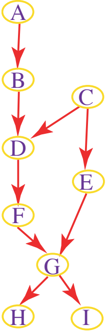


- $f_1(B) := \sum_{a \in dom(A)} P(A=a) \cdot P(B|A=a)$
- $f_2(B,D,E) := \sum_{c \in dom(C)} P(C=c) \cdot P(D|B,C=c) \cdot P(E|C=c)$

Variable elimination example

Compute $P(G|H = h_1)$. Elimination order: A, C, E, H, I, B, D, F

- $P(G, H) =$
  $\sum_{B,D,E,F,I} f_1(B) \cdot f_2(B, D, E) \cdot P(F|D) \cdot P(G|F, E) \cdot P(H|G) \cdot P(I|G)$
- Eliminate E:
  $P(G, H) = \sum_{B,D,F,I} f_1(B) \cdot f_3(B, D, F, G) \cdot P(F|D) \cdot P(H|G) \cdot P(I|G)$

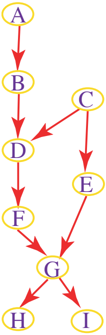


- $f_1(B) := \sum_{a \in dom(A)} P(A = a) \cdot P(B|A = a)$
- $f_2(B, D, E) := \sum_{c \in dom(C)} P(C = c) \cdot P(D|B, C = c) \cdot P(E|C = c)$
- $f_3(B, D, F, G) := \sum_{e \in dom(E)} f_2(B, D, E = e) \cdot P(G|F, E = e)$

Variable elimination example

Compute $P(G|H = h_1)$. Elimination order: A, C, E, H, I, B, D, F

- $P(G, H) = \sum_{B,D,F,I} f_1(B) \cdot f_3(B, D, F, G) \cdot P(F|D) \cdot P(H|G) \cdot P(I|G)$
- Observe $H = h_1$:
  $P(G, H = h_1) = \sum_{B,D,F,I} f_1(B) \cdot f_3(B, D, F, G) \cdot P(F|D) \cdot f_4(G) \cdot P(I|G)$

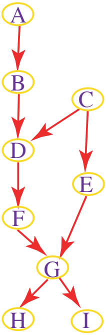


- $f_1(B) := \sum_{a \in dom(A)} P(A = a) \cdot P(B|A = a)$
- $f_2(B, D, E) := \sum_{c \in dom(C)} P(C = c) \cdot P(D|B, C = c) \cdot P(E|C = c)$
- $f_3(B, D, F, G) := \sum_{e \in dom(E)} f_2(B, D, E = e) \cdot P(G|F, E = e)$
- $f_4(G) := P(H = h_1|G)$

Variable elimination example

Compute $P(G|H=h_1)$. Elimination order: A, C, E, H, I, B, D, F

- $P(G, H=h_1) = \sum_{B,D,F,I} f_1(B) \cdot f_3(B,D,F,G) \cdot P(F|D) \cdot f_4(G) \cdot P(I|G)$
- Eliminate I:
  $P(G, H=h_1) = \sum_{B,D,F} f_1(B) \cdot f_3(B,D,F,G) \cdot P(F|D) \cdot f_4(G) \cdot f_5(G)$

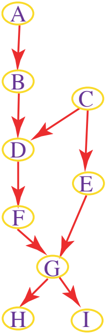


- $f_1(B) := \sum_{a \in dom(A)} P(A=a) \cdot P(B|A=a)$
- $f_2(B,D,E) := \sum_{c \in dom(C)} P(C=c) \cdot P(D|B,C=c) \cdot P(E|C=c)$
- $f_3(B,D,F,G) := \sum_{e \in dom(E)} f_2(B,D,E=e) \cdot P(G|F,E=e)$
- $f_4(G) := P(H=h_1|G)$
- $f_5(G) := \sum_{i \in dom(I)} P(I=i|G)$

Variable elimination example

Compute $P(G|H=h_1)$. Elimination order: A, C, E, H, I, B, D, F

- $P(G, H=h_1) = \sum_{B,D,F} f_1(B) \cdot f_3(B, D, F, G) \cdot P(F|D) \cdot f_4(G) \cdot f_5(G)$
- Eliminate B:
  $P(G, H=h_1) = \sum_{D,F} f_6(D, F, G) \cdot P(F|D) \cdot f_4(G) \cdot f_5(G)$

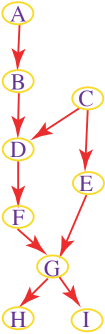


- $f_1(B) := \sum_{a \in dom(A)} P(A=a) \cdot P(B|A=a)$
- $f_2(B, D, E) := \sum_{c \in dom(C)} P(C=c) \cdot P(D|B, C=c) \cdot P(E|C=c)$
- $f_3(B, D, F, G) := \sum_{e \in dom(E)} f_2(B, D, E=e) \cdot P(G|F, E=e)$
- $f_4(G) := P(H=h_1|G)$
- $f_5(G) := \sum_{i \in dom(I)} P(I=i|G)$
- $f_6(D, F, G) := \sum_{b \in dom(B)} f_1(B=b) \cdot f_3(B=b, D, F, G)$

Variable elimination example

Compute $P(G|H=h_1)$. Elimination order: A, C, E, H, I, B, D, F

- $P(G, H=h_1) = \sum_{D,F} f_6(D,F,G) \cdot P(F|D) \cdot f_4(G) \cdot f_5(G)$
- Eliminate D: $P(G, H=h_1) = \sum_{F} f_7(F,G) \cdot f_4(G) \cdot f_5(G)$

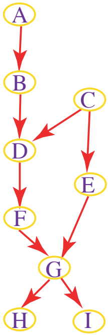


- $f_1(B) := \sum_{a \in dom(A)} P(A=a) \cdot P(B|A=a)$
- $f_2(B,D,E) := \sum_{c \in dom(C)} P(C=c) \cdot P(D|B,C=c) \cdot P(E|C=c)$
- $f_3(B,D,F,G) := \sum_{e \in dom(E)} f_2(B,D,E=e) \cdot P(G|F,E=e)$
- $f_4(G) := P(H=h_1|G)$
- $f_5(G) := \sum_{i \in dom(I)} P(I=i|G)$
- $f_6(D,F,G) := \sum_{b \in dom(B)} f_1(B=b) \cdot f_3(B=b,D,F,G)$
- $f_7(F,G) := \sum_{d \in dom(D)} f_6(D=d,F,G) \cdot P(F|D=d)$

Variable elimination example

Compute $P(G|H=h_1)$. Elimination order: A, C, E, H, I, B, D, F

- $P(G, H=h_1) = \sum_F f_7(F,G) \cdot f_4(G) \cdot f_5(G)$
- Eliminate F: $P(G, H=h_1) = f_8(G) \cdot f_4(G) \cdot f_5(G)$

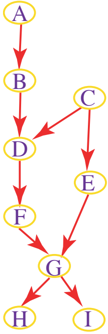


- $f_1(B) := \sum_{a \in dom(A)} P(A=a) \cdot P(B|A=a)$
- $f_2(B,D,E) := \sum_{c \in dom(C)} P(C=c) \cdot P(D|B,C=c) \cdot P(E|C=c)$
- $f_3(B,D,F,G) := \sum_{e \in dom(E)} f_2(B,D,E=e) \cdot P(G|F,E=e)$
- $f_4(G) := P(H=h_1|G)$
- $f_5(G) := \sum_{i \in dom(I)} P(I=i|G)$
- $f_6(D,F,G) := \sum_{b \in dom(B)} f_1(B=b) \cdot f_3(B=b,D,F,G)$
- $f_7(F,G) := \sum_{d \in dom(D)} f_6(D=d,F,G) \cdot P(F|D=d)$
- $f_8(G) := \sum_{f \in dom(F)} f_7(F=f,G)$

Variable elimination example

Compute $P(G|H=h_1)$. Elimination order: A, C, E, H, I, B, D, F

- $P(G, H=h_1) = f_8(G) \cdot f_4(G) \cdot f_5(G)$
- Normalize: $P(G|H=h_1) = \frac{P(G,H=h_1)}{\sum_{g \in dom(G)} P(G,H=h_1)}$



- $f_1(B) := \sum_{a \in dom(A)} P(A=a) \cdot P(B|A=a)$
- $f_2(B, D, E) := \sum_{c \in dom(C)} P(C=c) \cdot P(D|B, C=c) \cdot P(E|C=c)$
- $f_3(B, D, F, G) := \sum_{e \in dom(E)} f_2(B, D, E=e) \cdot P(G|F, E=e)$
- $f_4(G) := P(H=h_1|G)$
- $f_5(G) := \sum_{i \in dom(I)} P(I=i|G)$
- $f_6(D, F, G) := \sum_{b \in dom(B)} f_1(B=b) \cdot f_3(B=b, D, F, G)$
- $f_7(F, G) := \sum_{d \in dom(D)} f_6(D=d, F, G) \cdot P(F|D=d)$
- $f_8(G) := \sum_{f \in dom(F)} f_7(F=f, G)$

Lecture Overview

Recap

Variable Elimination Example

Hidden Markov Models

Markov chain

- A Markov chain is a special sort of belief network:

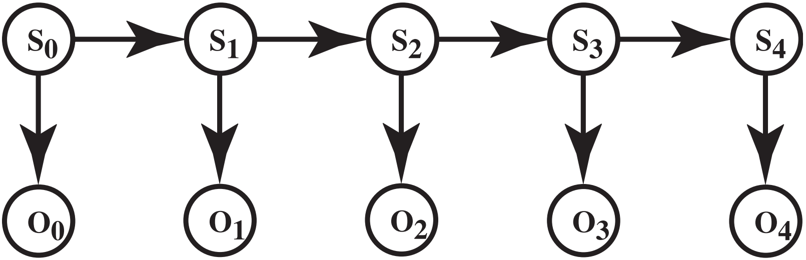
- Thus $P(S_{t+1}|S_0, \ldots, S_t) = P(S_{t+1}|S_t)$.
- Often S_t represents the state at time t. Intuitively S_t conveys all of the information about the history that can affect the future states.
- "The past is independent of the future given the present."

Stationary Markov chain

- A stationary Markov chain is when for all $t > 0$, $t' > 0$, $P(S_{t+1}|S_t) = P(S_{t'+1}|S_{t'})$.
- We specify $P(S_0)$ and $P(S_{t+1}|S_t)$.
  - Simple model, easy to specify
  - Often the natural model
  - The network can extend indefinitely

Hidden Markov Model

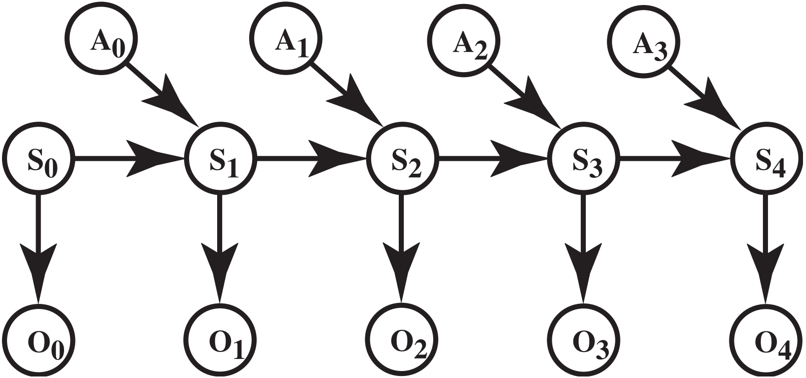
- A Hidden Markov Model (HMM) starts with a Markov chain, and adds a noisy observation about the state at each time step:



- $P(S_0)$ specifies initial conditions
- $P(S_{t+1}|S_t)$ specifies the dynamics
- $P(O_t|S_t)$ specifies the sensor model

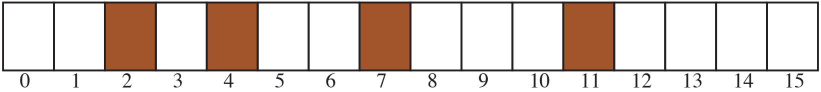
Example: localization

- Suppose a robot wants to determine its location based on its actions and its sensor readings: Localization
- This can be represented by the augmented HMM:



Example localization domain

- Circular corridor, with 16 locations:



- Doors at positions: 2, 4, 7, 11.
- Noisy Sensors
- Stochastic Dynamics
- Robot starts at an unknown location and must determine where it is.

Example Sensor Model

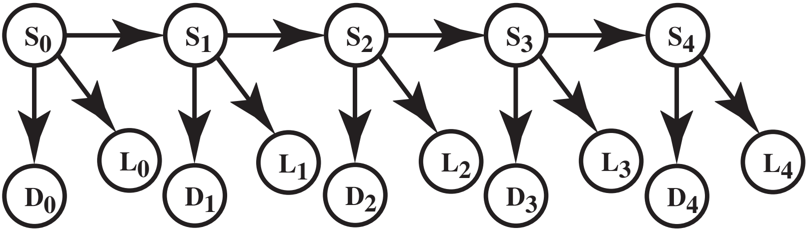
- $P(Observe\ Door \mid At\ Door) = 0.8$
- $P(Observe\ Door \mid Not\ At\ Door) = 0.1$

Example Dynamics Model

- $P(loc_{t+1} = L | action_t = goRight \wedge loc_t = L) = 0.1$
- $P(loc_{t+1} = L+1 | action_t = goRight \wedge loc_t = L) = 0.8$
- $P(loc_{t+1} = L+2 | action_t = goRight \wedge loc_t = L) = 0.074$
- $P(loc_{t+1} = L' | action_t = goRight \wedge loc_t = L) = 0.002$ for any other location L'.
  - All location arithmetic is modulo 16.
  - The action $goLeft$ works the same but to the left.

Combining sensor information

- Example: we can combine information from a light sensor and the door sensor Sensor Fusion



S_t robot location at time t
D_t door sensor value at time t
L_t light sensor value at time t

Localization demo

- http://www.cs.ubc.ca/spider/poole/demos/localization/localization.html