Reasoning Under Uncertainty: Introduction to Probability

CPSC 322 Lecture 23

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Textbook §9
Recap

Probability Introduction

Syntax and Semantics of Probability
Objects and Relations

- It is useful to view the world as consisting of objects and relationships between these objects.
- Often the propositions we spoke about before can be condensed into a much smaller number of propositions if they are allowed to express relationships between objects and/or functions of objects.
- Thus, reasoning in terms of objects and relationships can be simpler than reasoning in terms of features, as you can express more general knowledge using logical variables.
Syntax of Datalog

- **variable** starts with upper-case letter.
- **constant** starts with lower-case letter or is a sequence of digits (numeral).
- **predicate symbol** starts with lower-case letter.
- **term** is either a variable or a constant.
- **atomic symbol** (atom) is of the form \( p \) or \( p(t_1, \ldots, t_n) \) where \( p \) is a predicate symbol and \( t_i \) are terms.
Syntax of Datalog (cont)

- **definite clause** is either an atomic symbol (a fact) or of the form:

  \[ a \leftarrow b_1 \land \cdots \land b_m \]

  where \( a \) and \( b_i \) are atomic symbols.

- **query** is of the form \(?b_1 \land \cdots \land b_m\).

- **knowledge base** is a set of definite clauses.
Formal Semantics

An interpretation is a triple \( I = \langle D, \phi, \pi \rangle \), where

- \( D \), the domain, is a nonempty set. Elements of \( D \) are individuals.

- \( \phi \) is a mapping that assigns to each constant an element of \( D \). Constant \( c \) denotes individual \( \phi(c) \).

- \( \pi \) is a mapping that assigns to each \( n \)-ary predicate symbol a relation: a function from \( D^n \) into \{ \text{TRUE, FALSE} \}.
A constant $c$ denotes in $I$ the individual $\phi(c)$.

Ground (variable-free) atom $p(t_1, \ldots, t_n)$ is

- true in interpretation $I$ if $\pi(p)(t'_1, \ldots, t'_n) = TRUE$, where $t_i$ denotes $t'_i$ in interpretation $I$ and
- false in interpretation $I$ if $\pi(p)(t'_1, \ldots, t'_n) = FALSE$.

Ground clause $h \leftarrow b_1 \wedge \ldots \wedge b_m$ is false in interpretation $I$ if $h$ is false in $I$ and each $b_i$ is true in $I$, and is true in interpretation $I$ otherwise.
Variables

- Variables are **universally quantified** in the scope of a clause.
- A **variable assignment** is a function from variables into the domain.
- Given an interpretation and a variable assignment, each term denotes an individual and each clause is either true or false.
- A clause containing variables is true in an interpretation if it is true for all variable assignments.
Lecture Overview

Recap

Probability Introduction

Syntax and Semantics of Probability
Agents don’t have complete knowledge about the world.

Agents need to make decisions based on their uncertainty.

It isn’t enough to assume what the world is like.

**Example:** wearing a seat belt.

An agent needs to reason about its uncertainty.

When an agent makes an action under uncertainty, it is gambling $\Rightarrow$ probability.
Probability

- Probability is formal measure of uncertainty. There are two camps:
- **Frequentists**: believe that probability represents something *objective*, and compute probabilities by counting the frequencies of different events
- **Bayesians**: believe that probability represents something *subjective*, and understand probabilities as degrees of belief.
  - They compute probabilities by starting with *prior beliefs*, and then updating beliefs when they get new data.
  - *Example*: Your degree of belief that a bird can fly is your measure of belief in the flying ability of an individual based only on the knowledge that the individual is a bird.
  - Other agents may have different probabilities, as they may have had different experiences with birds or different knowledge about this particular bird.
  - An agent’s belief in a bird’s flying ability is affected by what the agent knows about that bird.
Numerical Measures of Belief

- Belief in proposition, $f$, can be measured in terms of a number between 0 and 1 — this is the **probability of $f$**.
  - The probability $f$ is 0 means that $f$ is believed to be definitely false.
  - The probability $f$ is 1 means that $f$ is believed to be definitely true.
- Using 0 and 1 is purely a convention.
- $f$ has a probability between 0 and 1, doesn’t mean $f$ is true to some degree, but means you are ignorant of its truth value. Probability is a measure of your ignorance.
Random Variables

- A random variable is a term in a language that can take one of a number of different values.
- The domain of a variable $X$, written $\text{dom}(X)$, is the set of values $X$ can take.
- A tuple of random variables $\langle X_1, \ldots, X_n \rangle$ is a complex random variable with domain $\text{dom}(X_1) \times \cdots \times \text{dom}(X_n)$. Often the tuple is written as $X_1, \ldots, X_n$.
- Assignment $X = x$ means variable $X$ has value $x$.
- A proposition is a Boolean formula made from assignments of values to variables.
Lecture Overview

Recap

Probability Introduction

Syntax and Semantics of Probability
Possible World Semantics

- A possible world specifies an assignment of one value to each random variable.
- $w \models X = x$ means variable $X$ is assigned value $x$ in world $w$.
- Logical connectives have their standard meaning:
  
  \[
  w \models \alpha \land \beta \text{ if } w \models \alpha \text{ and } w \models \beta \\
  w \models \alpha \lor \beta \text{ if } w \models \alpha \text{ or } w \models \beta \\
  w \models \neg \alpha \text{ if } w \not\models \alpha
  \]

- Let $\Omega$ be the set of all possible worlds.
Semantics of Probability: finite case

For a finite number of possible worlds:

- Define a nonnegative measure $\mu(w)$ to each world $w$ so that the measures of the possible worlds sum to 1.
- The measure specifies how much you think the world $w$ is like the real world.
- The probability of proposition $f$ is defined by:

$$P(f) = \sum_{w \models f} \mu(w).$$
Axioms of Probability: finite case

Four axioms define what follows from a set of probabilities:

**Axiom 1** \( P(f) = P(g) \) if \( f \leftrightarrow g \) is a tautology. That is, logically equivalent formulae have the same probability.

**Axiom 2** \( 0 \leq P(f) \) for any formula \( f \).

**Axiom 3** \( P(\tau) = 1 \) if \( \tau \) is a tautology.

**Axiom 4** \( P(f \lor g) = P(f) + P(g) \) if \( \neg(f \land g) \) is a tautology.

- These axioms are sound and complete with respect to the semantics.
Semantics of Probability: general case

In the general case we have a measure on sets of possible worlds, satisfying:

- \( \mu(S) \geq 0 \) for \( S \subseteq \Omega \)
- \( \mu(\Omega) = 1 \)
- \( \mu(S_1 \cup S_2) = \mu(S_1) + \mu(S_2) \) if \( S_1 \cap S_2 = \{\} \).