Scaling Up Game Theory: Representation and Reasoning with Action Graph Games

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This talk is primarily based on papers with:

Albert Xin Jiang [AAAI 2006]
Navin A.R. Bhat [UAI 2004]

and a joint paper [GEB, to appear 2010]

and also touches on more recent joint work with
Albert Xin Jiang, David R.M. Thompson,
Avi Pfeffer, Damien Bargiacchi, and James Wright

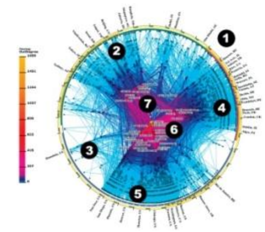
The Kind of Games Often Studied

- e.g., Prisoner's Dilemma: you and an accomplice are arrested. Should you confess or stay silent?

	C	D
C	$-5, -5$	$-20, 0$
D	$0, -20$	$-1, -1$

- The analysis of such 2×2 games has proven surprisingly interesting, and has had a profound impact both on our understanding of strategic situations and on popular culture

The Kind of Games We'd Like to Study



- In order to use game theory to model real systems, we need to consider games with more than two agents and two actions

- Some examples of the kinds of questions we would like to be able to answer:
  - How will heterogeneous users route their traffic in a network?
  - How will advertisers bid in a sponsored search auction?
  - Which job skills will students choose to pursue?
  - Where in a city will businesses choose to locate?

- Most GT work is analytic, not computational
- What's holding us back?
  - the size of classical game representations grows exponentially in the number of players
    - this makes all but the simplest games infeasible to write down
  - even when games can be represented, "fast" algorithms often have worst-case performance exponential in the game's size

Compact Representations

Research program for advancing the computational analysis of games:

1. find representations that can encode games of interest in exponentially-less space than the normal form
2. find efficient algorithms for working with these representations

Key representations from the literature:

- Graphical Games [Kearns, Littman, Singh, 2001]
  - utility functions exhibit strict independence
    - some pairs of agents have no (direct) effect on each other's payoff
  - many efficient algorithms
  - however, none of the games discussed above are compact as GGs
- Congestion Games [Rosenthal, 1973; Monderer & Shapley, 1996]
  - utility functions exhibit context-specific independence
    - whether agents affect each other's payoffs can depend on the action choices they each make
  - good theoretical properties; some algorithmic results
  - however, none of the games discussed above can be represented as CGs

Overview of This Talk

1. **Basic AGGs: Definition and Examples**
2. Analyzing and Extending the Representation
3. Computing Expected Utility
4. Recent Directions

The Coffee Shop Problem

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1075 Georgia Street West, Vancouver, BC V6E 3C9
(604) 683-1486

B. Melriches Coffeehouse
1244 Davie Street, Vancouver, BC V6E 1N3
(604) 689-5282

C. Hole In The Wall Cappuccino Bar
1030 Georgia Street West, Vancouver, BC V6E 2Y3
(604) 646-4653

D. Starbucks **Coffee** Co
1055 W Georgia, Vancouver, BC V5K 1A1
(604) 685-5882

E. Five Roses Bakery Cafe
1220 Bute Street, Vancouver, BC V6E 1Z8
(604) 669-8989

F. Starbucks **Coffee** Co
1095 Howe Street, Vancouver, BC V6Z 1P6
(604) 685-7083

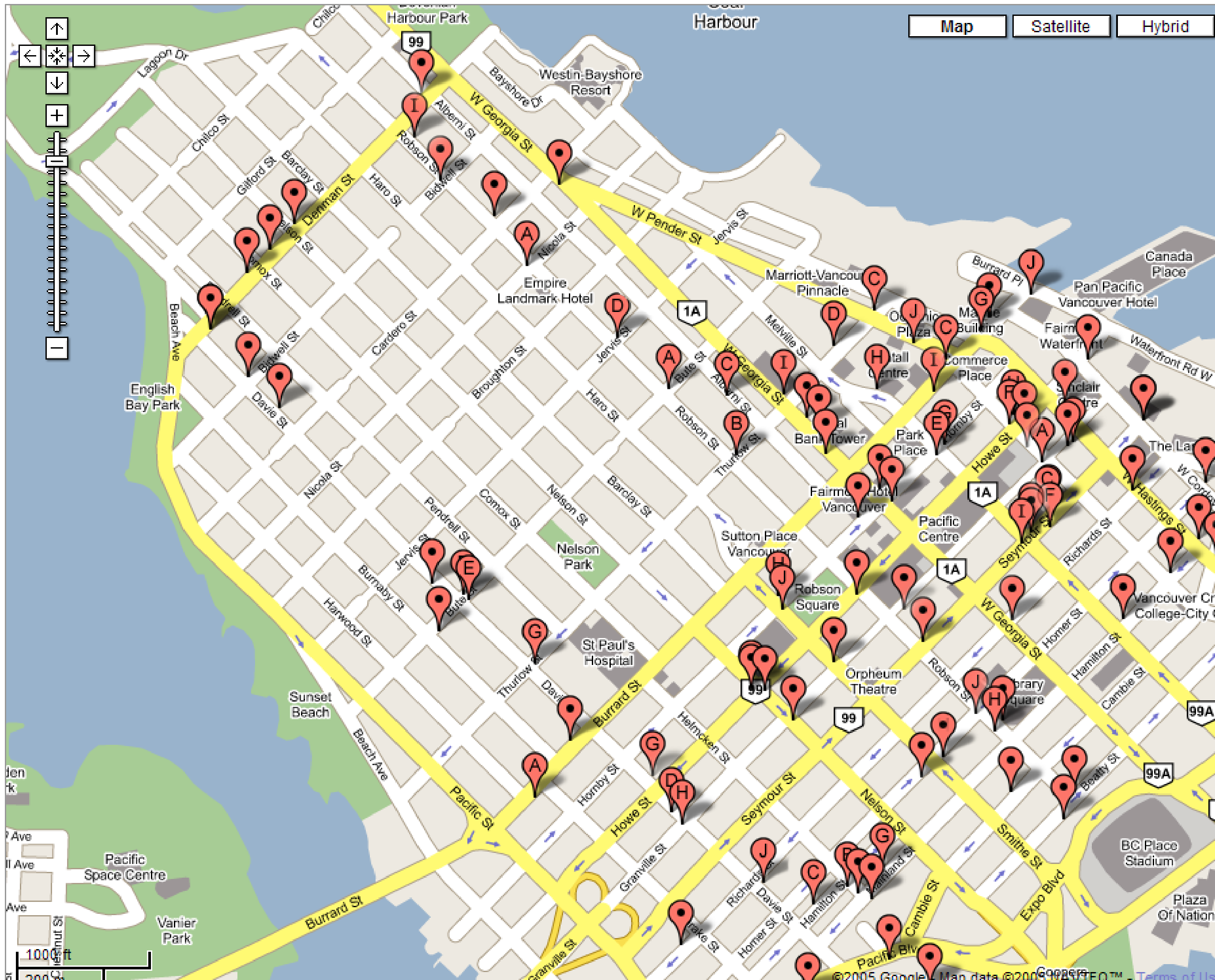
G. Uptown Espresso
808 Nelson Street, Vancouver, BC V6Z 2H2
(604) 689-1920

H. Caffe Artigiano
763 Hornby Street, Vancouver, BC V6Z 1S2
(604) 696-9222

I. Skyline Expresso
900 Howe Street, Vancouver, BC V6Z 2M4
(604) 683-4234

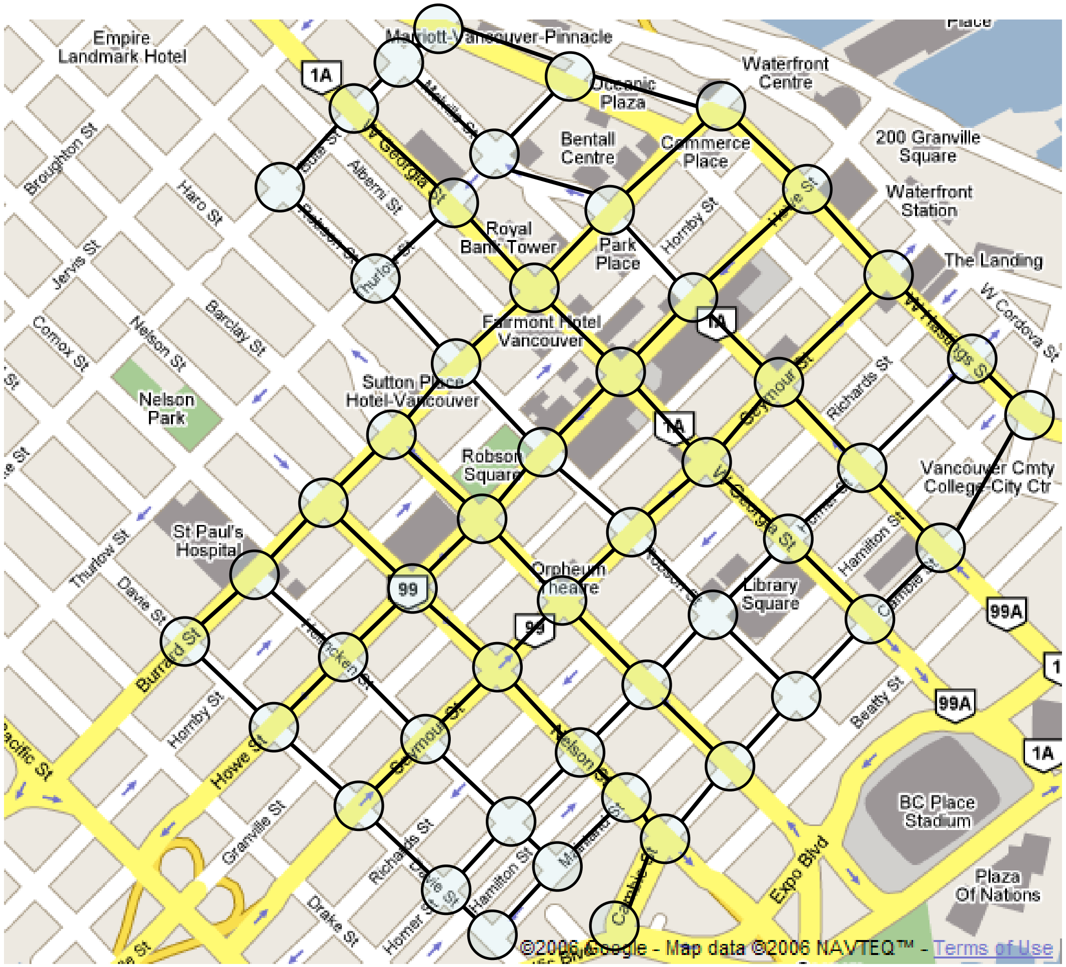
J. Farenheit Celsius **Coffee**
1225 Burrard Street, Vancouver, BC V6Z 1Z5
(604) 682-6675

Chicco Dall Oriente
1504 Robson Street, Vancouver, BC V6G 1C2



Basic Action-Graph Games

- set of **players:** want to open coffee shops
- **actions:** choose a location for your shop, or choose not to enter the market
- **utility:** profitability of a location
  - some locations might have more customers, and so might be better *ex ante*
  - utility also depends on the number of other players who choose the same or an adjacent location



Formal Definitions

Definition 1 (action graph) *An* **action graph** *is a tuple* $(\mathcal{A}, E)$, *where* $\mathcal{A}$ *is a set of nodes corresponding to distinct actions and* E *is a set of directed edges.*

Let $A = A_1 \times \cdots \times A_n$ be a ***set of actions*** available to each of n agents, with $\mathcal{A} = \bigcup_{i \in N} A_i$.

Definition 2 (configuration) *Given an action graph* $(\mathcal{A}, E)$ *and a set of action profiles* A, *a* **configuration** c *is a tuple of* $|\mathcal{A}|$ *non-negative integers, where the* j^{th} *element* $c[j]$ *is interpreted as the number of agents who chose the* j^{th} *action* $a_j \in \mathcal{A}$, *and where there exists some* $a \in A$ *that would give rise to* c. *Denote the set of all configurations as* C.

Formal Definitions

Definition 3 (neighborhood relation) *Given a graph having a set of nodes $\mathcal{A}$ and edges E, define the* **neighborhood relation** *as $\nu : \mathcal{A} \rightarrow 2^{\mathcal{A}}$, with $\nu(i) = \{j | (j, i) \in E\}$.*

Define a ***configuration over a node's neighborhood***, written as $c^{(\alpha)} \in C^{(\alpha)}$, as the elements of c that correspond to the actions $\nu(\alpha)$.

Definition 4 *A* **basic action-graph game (AGG-$\emptyset$)** *is a tuple (N, A, G, u):*

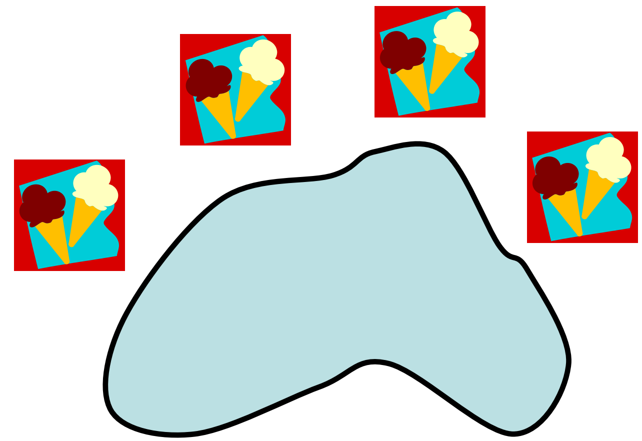
- *N is the set of agents;*
- *$A = A_1 \times \cdots \times A_n$, where A_i is the set of actions available to agent i;*
- *$G = (\mathcal{A}, E)$ is an action graph, where $\mathcal{A} = \bigcup_{i \in N} A_i$ is the set of distinct actions;*
- *$u = (u^1, \ldots, u^{|\mathcal{A}|})$, $u^{\alpha} : C^{(\alpha)} \rightarrow \mathbb{R}$.*

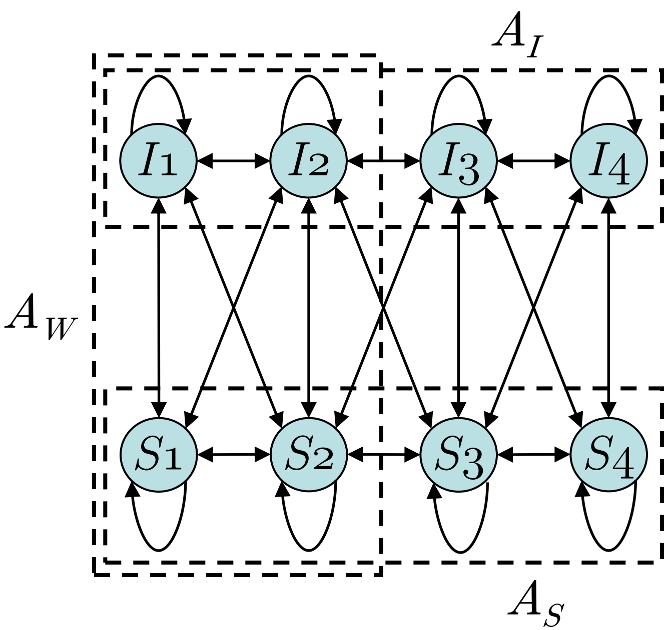
Elaborated Ice Cream Vendor Problem

Inspired by [Hotelling, 1929]

n vendors sell either ice cream or strawberries at one of four stations along a beach

- n_I ice cream (I) vendors;
- n_S strawberry (S) vendors;
- n_W can sell I/S, but only on the west side.
- **competition** between nearby sellers of same type; **synergy** between nearby different types





Notes:

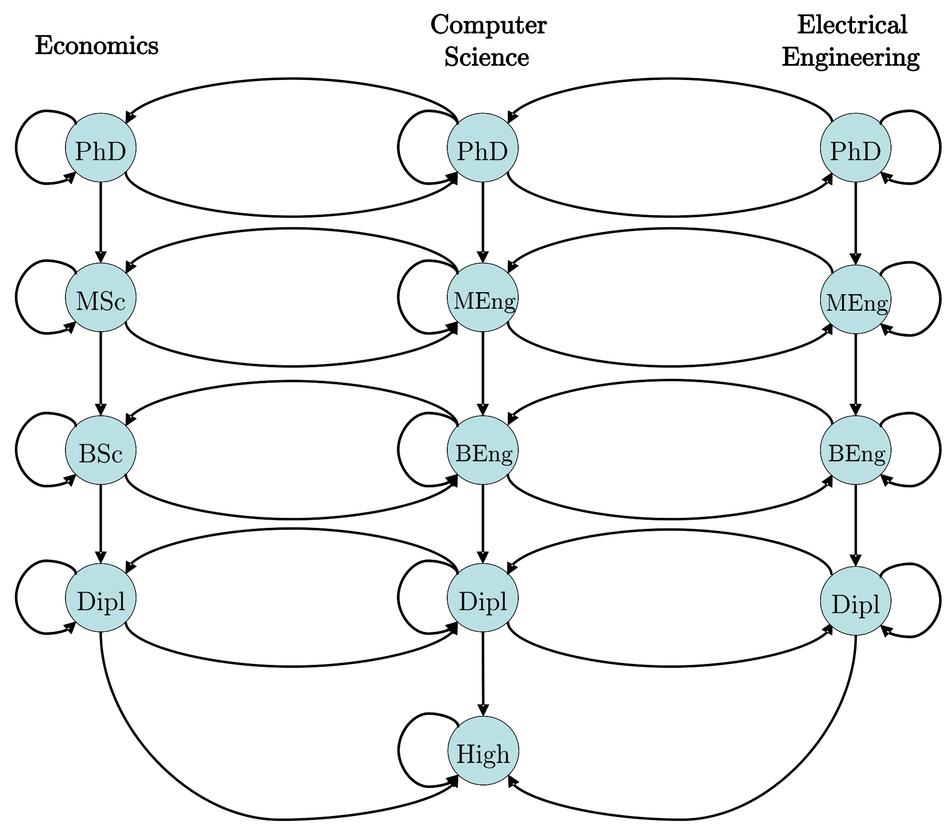
- graph structure independent of # agents
- overlapping action sets
- context-specific independence without strict independence

The Job Market Problem

Each player chooses a level of training

Players' utilities are the sum of:

- a constant cost:
  - difficulty; tuition; foregone wages
- a variable reward, depending on:
  - How many jobs prefer workers with this training, and how desirable are the jobs?
  - How many other jobs are willing to take such workers as a second choice, and how good are these jobs?
    - Employers will take workers who are overqualified, but only by one degree.
    - They will also interchange similar degrees, but only at the same level.
  - How many other graduates want the same jobs?



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Analyzing the AGG-∅ Representation

AGG-∅s can represent **any game**.

Overall, AGG-∅s are **more compact than the normal form** when the game exhibits either or both of the following properties:

1. **Context-Specific Independence**:
   - pairs of agents can choose actions that are not neighbors in the action graph

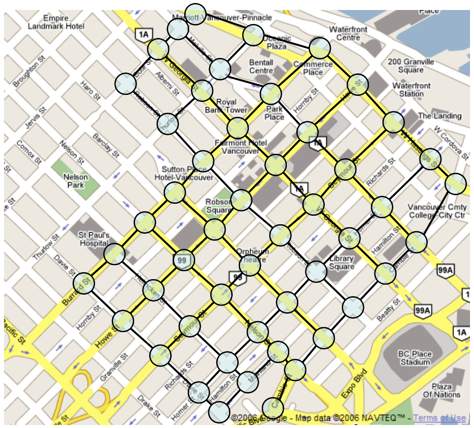
2. **Anonymity**:
   - multiple action profiles yield the same configuration

When max in-degree $\mathcal{I}$ is bounded by a constant:

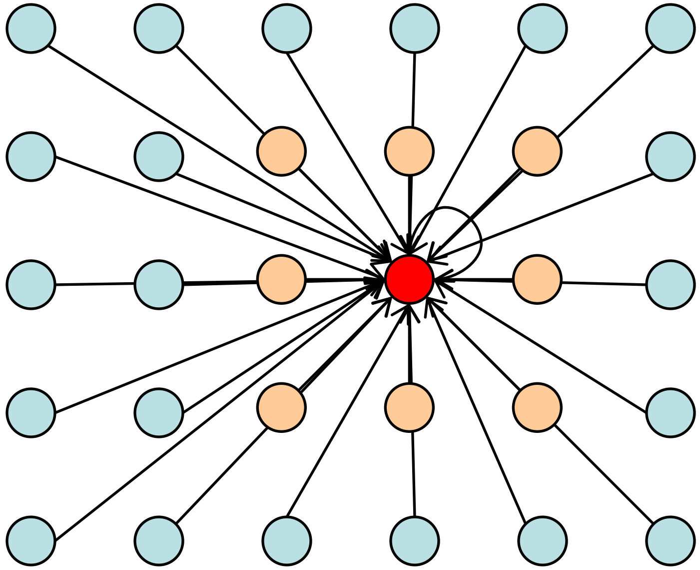
– **polynomial size:** $O(|A_{\max}|n^{\mathcal{I}})$

– in contrast, size of normal form is $\mathrm{O}(n|A_{\max}|^{n})$

The Coffee Shop Problem Revisited



- What if utility also depends on total # shops?
- Now action graph has in-degree $|\mathcal{A}|$
  - NF & Graphical Game representations: $O(|\mathcal{A}|^N)$
  - AGG-∅ representation: $O(N^{|\mathcal{A}|})$
  - when $|\mathcal{A}|$ is held constant, the AGG-∅ representation is polynomial in N
    - but still doesn't effectively capture game structure
    - given i's action, his payoff depends only on 3 quantities!

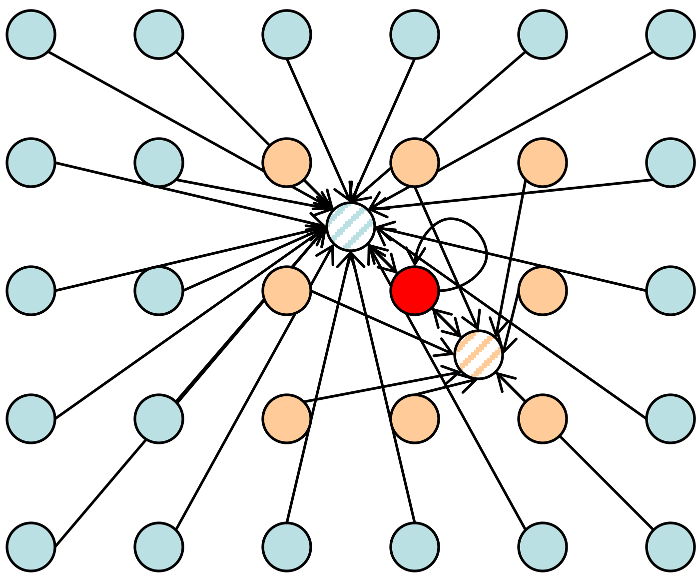


6 × 5 Coffee Shop Problem: projected action graph at the red node

AGG-FNs: Function Nodes

- To exploit this structure, introduce **function nodes**:
  - The "configuration" of a function node p is a (given) function of the configuration of its neighbors: $c[p] = f_p(c[\nu(p)])$

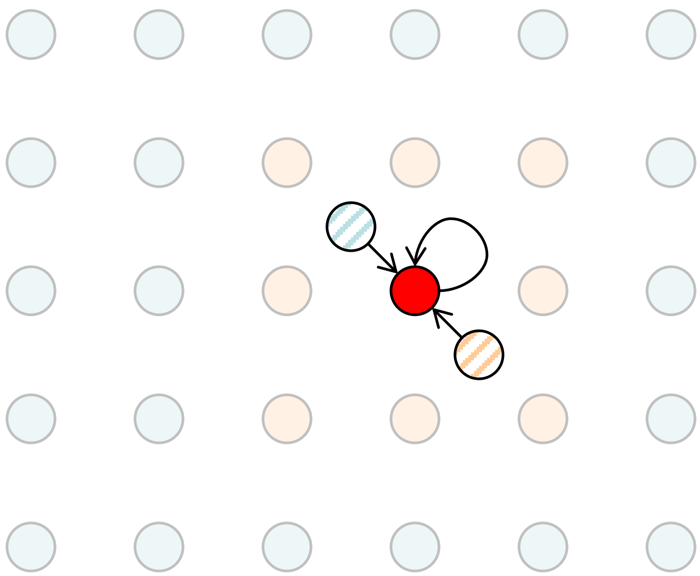
- **Coffee-shop example**: for each action node s, introduce:
  - a function node with adjacent actions as neighbors
    - $c[p'_s]$ = total number of shops in surrounding nodes
  - similarly, a function node with non-adjacent actions as neighbors



6 × 5 Coffee Shop Problem: function nodes for the red node

The Coffee Shop Problem

- Now the red node has only three incoming edges:
  - itself, the blue function node and the orange function node
  - so, the action-graph now has in-degree three
- Size of representation is now $O(N^3)$

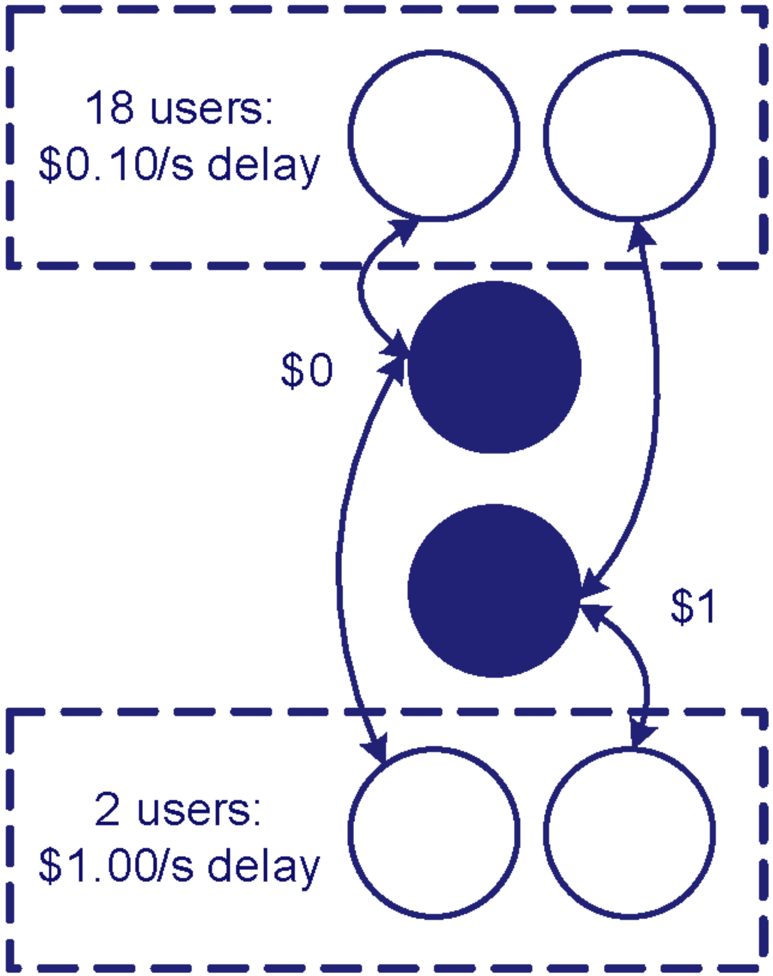


6×5 Coffee Shop Problem: projected action graph at the red node

Example: Parallel Edges

Based on [Thompson, Jiang & LB, 2007]; inspired by [Odlyzko, 1998]

- Network with one source, one sink, two parallel edges
  - both edges offer identical speed
  - one is free, one costs $1
  - latency is an additive function of the number of users on an edge
- Two classes of users
  - 18 users pay $0.10/unit of delay
  - 2 users pay $1.00/unit of delay
- Which edge should users choose?
- Example scales to longer paths
  - not a congestion game because of player-specific utility



Further Representational Results

- Without loss of compactness, AGGs can also encode:
  - Graphical games (AGG-∅)
  - Symmetric games (AGG-∅)
  - Anonymous games (AGG-FN)

- One other extension to AGGs: explicit **additive structure**
- Enables compact encoding of still other game classes:
  - Congestion games (AGG-FNA)
  - Polymatrix games (AGG-FNA)
  - Local-Effect games (AGG-FNA)

Conclusion: AGGs compactly encode **all major compact classes** of simultaneous-move games, and also **many new games** that are compact in none of these representations.

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Computing Expected Utility

Expected utility of agent i for playing (pure) action a_i, if other agents play according to mixed-strategy profile s_{-i}:

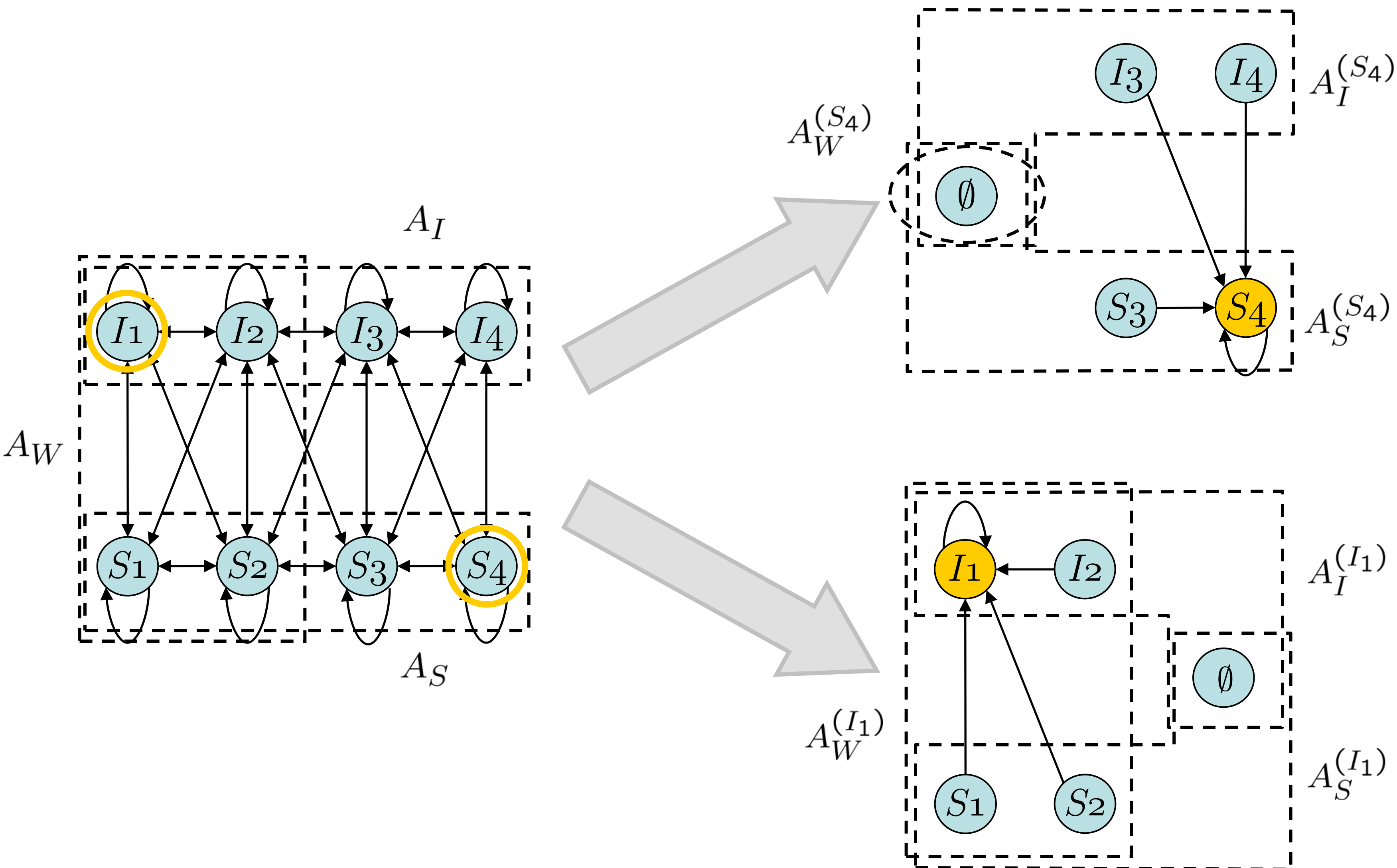
$$V_{a_i}^i(s_{-i}) \equiv \sum_{a_{-i} \in A_{-i}} u_i(a_i, a_{-i}) \, Pr(a_{-i} | s_{-i})$$

Exponential-sized set: naive algorithm is $O\left(|A_{\max}|^{n-1}\right)$

$V_{a_i}^i(s_{-i})$ is an inner-loop problem in many game-theoretic algorithms:

- Best Response (e.g., for multiagent reinforcement learning)
- Govindan-Wilson Algorithm (Nash equilibrium)
- Simplicial Subdivision Algorithm (Nash equilibrium)
- Papadimitriou's Algorithm (correlated Nash equilibrium)
- Turocy's Path Tracing Algorithm (quantal response equilibrium)
- Predicted Action Distributions under Level-k; Cognitive Hierarchy

Computing with AGG-∅s: Projection



Computing with AGG-∅s: Projection

- Projection captures context-specific independence and strict independence

$$V^i_{a_i}(s_{-i}) = \sum_{a_{-i}^{(a_i)} \in A_{-i}^{(a_i)}} u^{a_i}\left(\mathcal{C}(a_i, a_{-i}^{(a_i)})\right) Pr\left(a_{-i}^{(a_i)} | s_{-i}^{(a_i)}\right)$$

Still exponential, but smaller than before

$$Pr\left(a_{-i}^{(a_i)} | s_{-i}^{(a_i)}\right) = \prod_{j \in N \setminus \{i\}} s_j^{(a_i)}(a_j^{(a_i)}).$$

Linear-sized set

$*^{(\alpha)} \equiv$ projection with respect to action α

$\mathcal{C}(a_i, a_{-i}) \equiv$ configuration caused by a_i, a_{-i}

$\mathcal{S}(c) \equiv$ set of pure action profiles giving rise to c

Computing with AGG-∅s: Anonymity

- Writing in terms of the configuration captures anonymity

$$V^i_{a_i}(s_{-i}) = \sum_{c^{(a_i)}_{-i} \in C^{(a_i)}_{-i}} u^{a_i}\left(\mathcal{C}\left(a_i, c^{(a_i)}_{-i}\right)\right) Pr\left(c^{(a_i)}_{-i} | s^{(a_i)}_{-i}\right)$$

Polynomial-sized set

$$Pr\left(c^{(a_i)}_{-i} | s^{(a_i)}_{-i}\right) = \sum_{a^{(a_i)}_{-i} \in \mathcal{S}\left(c^{(a_i)}_{-i}\right)} Pr\left(a^{(a_i)}_{-i} | s^{(a_i)}_{-i}\right)$$

Exponential-sized set

$*^{(\alpha)} \equiv$ projection with respect to action α

$\mathcal{C}(a_i, c_{-i}) \equiv$ configuration caused by a_i, c_{-i}

$\mathcal{S}(c) \equiv$ set of pure action profiles giving rise to c

Dynamic Programming

- Can we **do better** computing $Pr\left(c_{-i}^{(a_i)} | s_{-i}^{(a_i)}\right)$? Note that
  - the players' mixed strategies are independent
    - s is a product probability distribution
  - each player affects a configuration c independently

- We can use **dynamic programming** to compute the probability of a configuration:
  - base case: zero agents and the mixed strategy s_0:
    - $C_0 = \{c_0\}$
    - $c_0 = [0, \ldots, 0]$
    - $P_0(c_0) = 1$
  - then add agents **one by one**:
    - C_k: the set of configurations that can be built by adding any action from the support of player k's mixed strategy to any configuration from $C_{k\text{-}1}$
    - $P_k(c_k) = \displaystyle\sum_{\substack{(c_{k-1}, a_k),\\ \mathcal{C}(c_{k-1}, a_k) = c_k}} s_k(a_k) \cdot P_{k-1}(c_{k-1})$

Computing with AGGs: Complexity

Theorem 1 *Given an AGG-$\emptyset$ representation of a game, i's expected payoff $V^i_{a_i}(s_{-i})$ can be computed in time polynomial in the size of the representation. If $\mathcal{I}$, the maximum in-degree of the action graph, is bounded by a constant, $V^i_{a_i}(s_{-i})$ can be computed in time polynomial in n.*

- Complexity of our approach:
  $$O\left(n^{\mathcal{I}} poly(n) poly\left(|A_{\max}|\right)\right)$$
- Exponential speedup vs. standard approach:
  $$O\left(|A_{\max}|^{n-1} poly(n) poly(|A_{\max}|)\right)$$

In AGG-FNs, players are no longer guaranteed to affect c independently

- but the DP algorithm still works when function nodes can be expressed using some commutative, associative operator

Computing Expected Utility

$V^i_{a_i}(s_{-i})$ is an inner-loop problem in many game-theoretic algorithms:

- Best Response (e.g., for multiagent reinforcement learning)
- Govindan-Wilson Algorithm (Nash equilibrium)
- Simplicial Subdivision Algorithm (Nash equilibrium)
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Because we compute $V^i_{a_i}(s_{-i})$ exactly, our expected utility algorithm yields an exponential speedup in every one of these algorithms, whenever the AGG is exponentially smaller than the normal form.

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1. Basic AGGs: Definition and Examples
2. Analyzing and Extending the Representation
3. Computing Expected Utility
4. **Recent Directions**
    1. computing pure strategy equilibria
    2. analyzing sponsored search auctions
    3. temporal AGGs
    4. Bayesian AGGs
    5. free software tools

(1) Computing Pure-Strategy Equilibrium

- Pure Nash equilibrium is often a more interesting solution concept than mixed Nash equilibrium

- It also presents a very computationally different problem
  - PSNE in normal form admits a very simple polytime algorithm
    - just check every action profile
  - For AGG-∅s the representation can be exponentially smaller
    - thus, the same algorithm is exponential time

Theorem (Conitzer, personal communication; also proven independently in (Daskalakis et al. 2008)): The problem of determining whether a pure Nash equilibrium exists in an AGG-∅ is NP-complete, even when the AGG-∅ is symmetric and has max in-degree of three.

(1) Computing PSNEs in AGG-∅s

[Jiang & LB, 2007]

We propose a **message passing algorithm**:

- partition action graph into subgraphs (via tree decomposition)
- construct equilibria of the game from equilibria of games played on subgraphs

This algorithm finds PSNE in polynomial time for every **symmetric AGG-∅ that has bounded treewidth.**

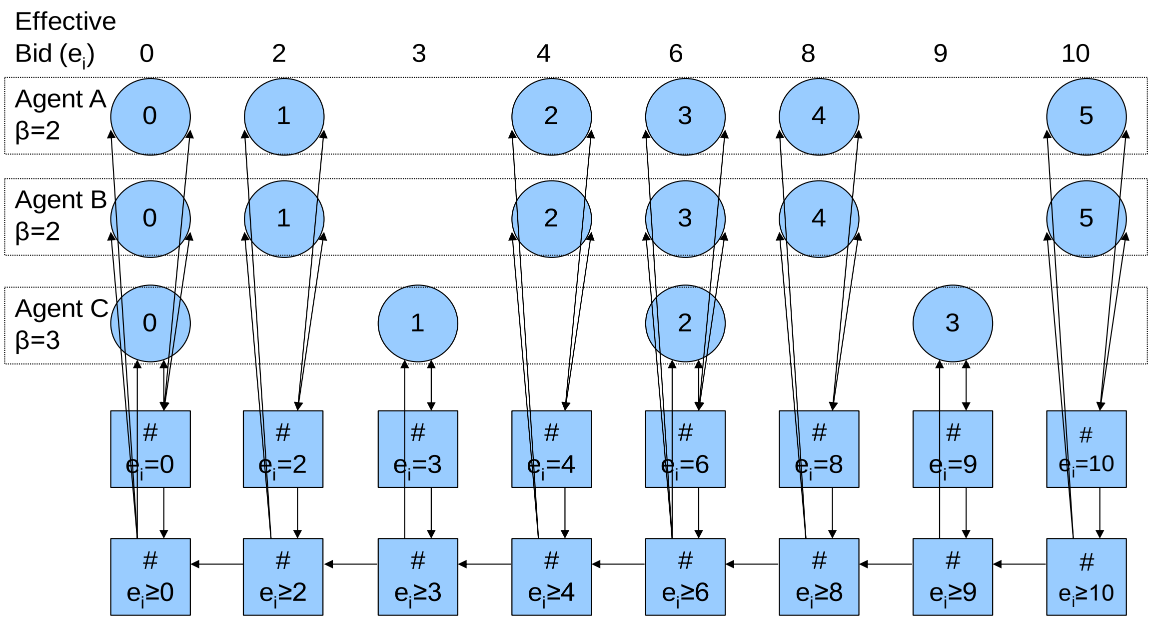
- it can also be applied to other bounded-treewidth settings

- Generalizes earlier algorithms
  - finding pure equilibria in **graphical games**
    [Gottlob, Greco, & Scarcello 2003; Daskalakis & Papadimitriou 2006]
  - finding pure equilibria in **simple congestion games**
    [Ieong, McGrew, Nudelman, Shoham, & Sun 2005]

(2) Sponsored Search Auctions

[Thompson & LB, 2008; 2009]

- Position auctions are used to sell $10Bs of keyword ads
- Some theoretical analysis, but based on strong assumptions
  - Unknown how different auctions compare in more general settings
- Idea: analyze the auctions computationally
  - Main hurdle: ad auction games are large; infeasible as normal form



AGG-FN representation of a Weighted, Generalized First-Price (GFP) Auction

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[Thompson & LB, 2008; 2009]

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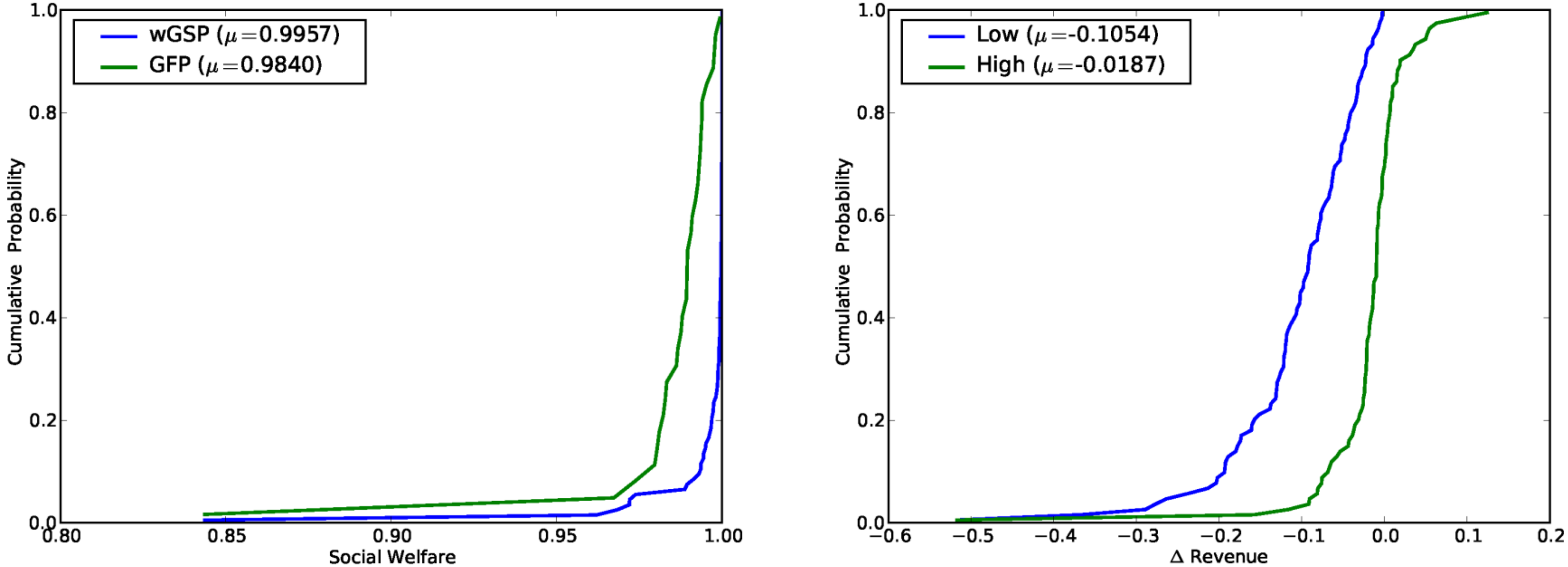


Social welfare and revenue of EOS auction model

(3) Temporal Action Graph Games

[Jiang, LB & Pfeffer, 2009]

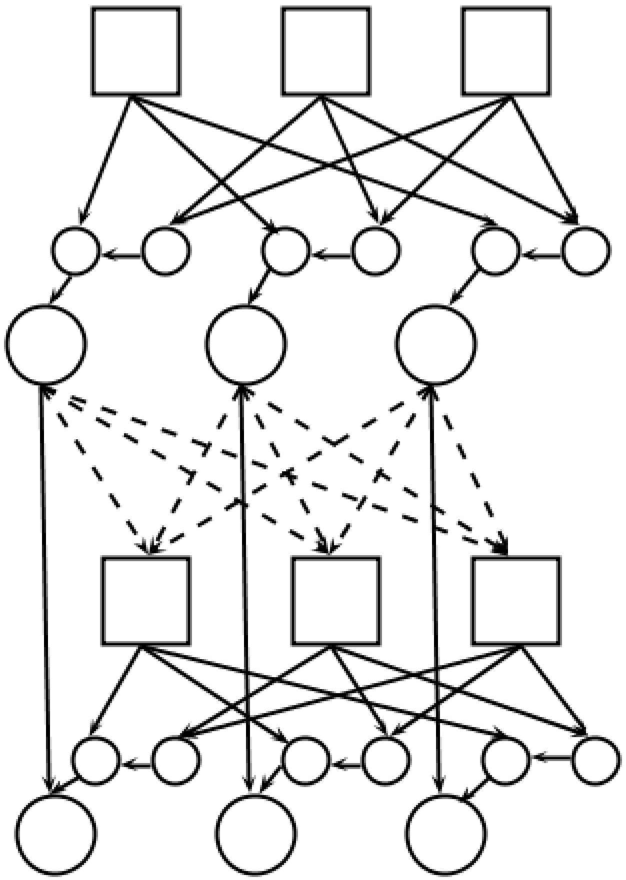
Goal: extend AGGs to **temporal settings**

- Model: An AGG-FN played over a series of **discrete time steps**
  - at each time step, a subset of players move
  - action counts on the action nodes grow over time
- Allow payoff uncertainty using **random variables** that are realized at a given time step
- Imperfect information: players may **condition their actions** on a given set of observed previous actions, chance variables and action counts
- Utility functions: action-specific and time-specific

(3) Properties of TAGGs

[Jiang, LB & Pfeffer, 2009]

- Can **compactly represent** a wide range of dynamic games, including:
  - arbitrary MAIDs [Milch & Koller, 2001]
  - games whose straightforward MAID representations are not compact
- Can be **efficiently encoded as MAIDs** by introducing deterministic chance nodes
- Efficient computation of **expected utility**
  - exploit anonymity and context-specific independence as in AGG-∅s
  - also exploit the temporal structure
  - as with AGG-∅s, can be leveraged to yield **exponential speedups in computation** (Nash equilibrium, etc.)



(4) Bayesian Games

- TAGGs aren't the most appropriate way of representing simultaneous-move Bayesian games
  - indeed, while such models are widely used (e.g., in auction theory), the setting has largely been neglected by the computational game theory community
- As far as we know, there are no representations or algorithms targeting general BNE computation
- This leaves two general approaches, both of which make use of complete-information Nash algorithms:
  1. Induced normal form
     - one action for each pure strategy (mapping from type to action)
     - set of players unchanged
  2. Agent form
     - one player for each type of each of the BG's players
     - action space unchanged

(4) Bayesian AGGs

[Jiang & LB, work under review 2010]

Bayesian AGG: an AGG-like representation of a Bayesian game's utility functions, which compactly encodes its agent form:

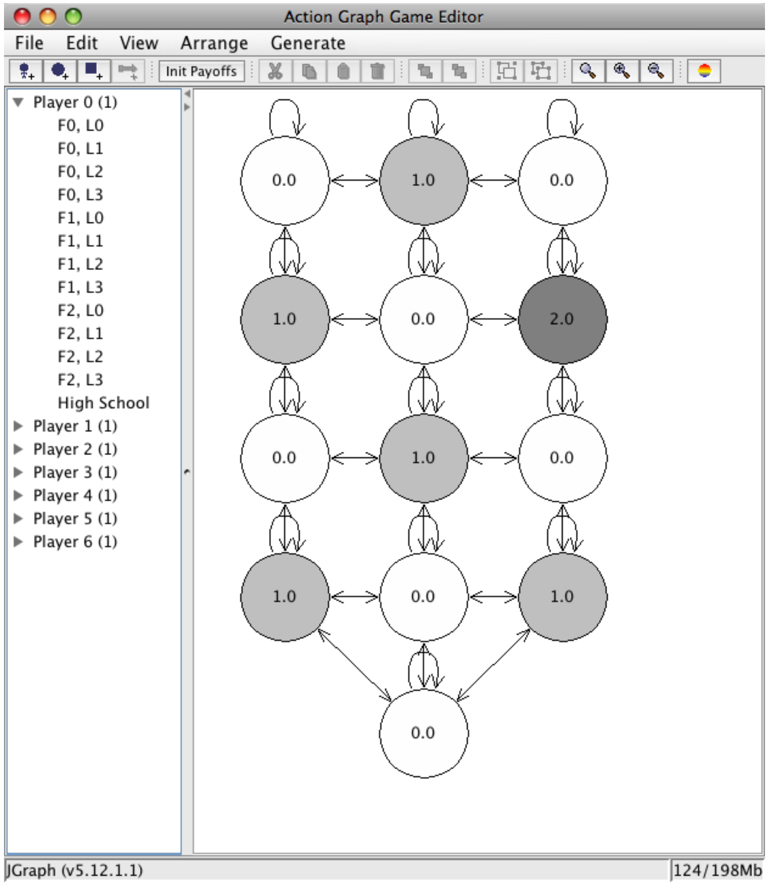
- **Bayesian network** for the joint type distribution
- A (potentially separate) **action graph** for each type of each agent
- A **utility function** that depends on which types are realized and on the actions taken by the other agents of the appropriate types

- **Representation size** grows polynomially in $|\Theta|$, $|A|$, n, when action graph has constant-bounded in-degree
  - Exponential savings over an unstructured Bayesian game
- When types are independent, expected utility can be **computed in time polynomial in the size of the BAGG**
- When types are not independent, expected utility can still be **computed in polynomial time** when an induced Bayesian network has bounded treewidth.

(5) Free Software Tools

[Jiang, Bargiacchi & LB, 2007–2010]

- Goal: make it easier for other researchers to use AGGs
- Equilibrium computation algorithms:
  - Govindan-Wilson (NE)
  - Simplicial Subdivision (NE)
  - Papadimitriou (CE) ** in progress*
  - Turocy (QRE) ** in progress*
- GAMUT:
  - extended to support AGGs
- Action Graph Game Editor:
  - creates AGGs graphically
  - facilitates entry of utility fns
  - supports "player classes"
  - auto creates game generators
  - visualizes eq. on the action graph



Conclusions

- **AGGs compactly represent games** exhibiting context-specific independence, anonymity and/or additive structure
- **Generalizes all major, existing compact representations** of simultaneous-move games
  - graphical games, congestion games, many others
- Recent directions:
  - Polytime algorithm for computing **pure strategy Nash equilibrium** (bounded treewidth; symmetric AGG-∅)
  - modeling and comparing **sponsored search auctions**
  - extending AGGs to **temporal settings**
  - extending AGGs to **Bayesian games**
  - developing **free software tools**

http://www.cs.ubc.ca/~kevinlb google://“Kevin Leyton-Brown”