# Revenue Monotonicity in Combinatorial Auctions

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Joint work with Baharak Rastegari and Anne Condon
Based on a AAAI-2007 paper, with stronger results

*Thanks to David Parkes, Ron Lavi and Daniel Lehmann for helpful discussions here at Dagstuhl!*

# Outline

# Setting

- $G$, a set of $m$ goods for sale
- $N = \{1, 2, \ldots, n\}$, the universal set of $n$ bidders
  - each may or may not participate in a given auction
- Each bidder $i$ has a valuation $v_i : 2^G \rightarrow \mathbb{R}^+ \cup \{0\}$.
- A deterministic, direct combinatorial auction (CA) mechanism:
  - asks each bidder $i$ to declare her valuation function $\widehat{v}_i$
  - allocates to $i$ the bundle $a_i(\widehat{v})$
  - requires $i$ to pay $p_i(\widehat{v})$
- The revenue of a CA mechanism is the sum of the payments made by the bidders, $R = \sum_{i \in N} p_i(\widehat{v})$.

# Revenue Monotonicity

- From single-good auctions, we have the intuition that adding bidders means more competition, and hence more revenue for the auctioneer.
  - Does this intuition extend to combinatorial auctions?

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A CA mechanism is revenue monotonic if dropping a bidder never increases the auction's revenue.

**Definition (Revenue Monotonicity)**

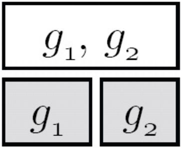
CA mechanism $M$ is revenue monotonic (RM) if for all $\hat{v}$ in the equilibrium of the mechanism and for all bidders $j$,

$$\sum_{i \in N} p_i(\hat{v}) \geq \sum_{i \in N \setminus \{j\}} p_i(\hat{v}_{-j}).$$

# VCG is not Revenue Monotonic

## Example (see e.g., [Ausubel & Milgrom, 2006])

| Bidder | $v(g_1)$ | $v(g_2)$ | $v(g_1, g_2)$ | $SW_{-i}$ without $i$ | $SW_{-i}$ with $i$ | $i$ pays |
|---|---|---|---|---|---|---|
| 1 | 11 | 0 | 11 | | | |
| 2 | 0 | 0 | 10 | | | |
| 3 | 0 | 11 | 11 | | | |



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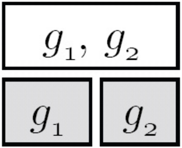
|---|---|---|---|---|---|---|
| 1 | 11 | 0 | 11 | 11 | 11 | 0 |
| 2 | 0 | 0 | 10 | 22 | 22 | 0 |
| 3 | 0 | 11 | 11 | 11 | 11 | 0 |

1 and 3 win

Revenue $= 0$



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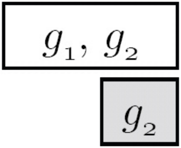

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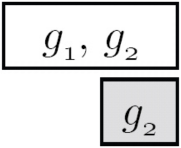
| 2 | 0 | 0 | 10 | 11 | 11 | 0 |
| 3 | 0 | 11 | 11 | 10 | 0 | 10 |

3 wins

Revenue $= 10$



# Plan of this talk

We are interested in whether this pathological revenue behavior can be avoided under other CA mechanisms.

- By the revelation principle, we can restrict attention to direct mechanisms.

In the rest of the talk I'll:

- Discuss desirable properties for CA mechanisms, including efficiency and relaxations
- Discuss single-mindedness and criticality
- Show that *no* deterministic, direct CA mechanism that satisfies our properties is revenue monotonic.
- Consider some consequences of this result.

# Outline

# Properties

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**Definition (Consumer sovereignty)**

For every bidder $i$, every set of bids $\widehat{v}_{-i}$, and every bundle $b_i$ there exists some finite amount $k_i \in \mathbb{R}$ such that if $i$ declares a value of $k_i$ for every bundle $b_i' \supseteq b_i$ and $0$ for all other bundles, $i$ is allocated at least $b_i$.

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## Definition (Efficiency)

The chosen allocation maximizes the social welfare, $\sum_i v_i(a_i)$.

# Efficiency Reconsidered

Efficiency is a very strong condition for us to require.

**Theorem (Green & Laffont, 1977)**

*The only DS truthful and efficient CA mechanisms are Groves mechanisms.*

We've already seen that VCG fails RM. Let's expand our search for RM mechanisms to a broader class that still includes efficient mechanisms.

# Efficiency Reconsidered

RM is unsatifyingly easy to achieve if we simply drop efficiency.

## Proposition

*The following bundling mechanism satisfies DS truthfulness, participation, consumer sovereignty and revenue monotonicity:*

1. *bundle all the goods together;*
2. *sell this bundle to the highest bidder;*
3. *charge this bidder the price offered by the second-highest bidder.*

Unless we're content with CA mechanisms like this one, we need to require something *like* efficiency, but weaker...

# Maximality

A mechanism is maximal with respect to a bidder $i$ if, whenever $i$'s valuation is sufficiently high, it never chooses allocations that could be augmented to satisfy $i$.

## Definition (Maximality)

A CA mechanism $M$ is maximal with respect to bidder $i$ iff $\forall s \subseteq 2^G$, there exists a nonnegative constant $\alpha_{i,s}$ such that $M$ always chooses an allocation where either:

- $v_i(a_i(\widehat{v})) > 0$; or
- the allocation cannot be augmented to award $i$ a bundle $s$ for which $v_i(s) > \alpha_{i,s}$.

- $\alpha_{i,s}$ is sort of like a bidder/bundle-specific reserve price.
- a weakening of the "reasonableness" condition of [Nisan & Ronen, 2000]

# Many interesting mechanisms are maximal

For example, a CA mechanism is maximal if the chosen allocation:

- ...is efficient
  - as before: VCG; other Groves mechanisms
- ...is strongly Pareto efficient[1]
  - the allocation cannot be changed to make some bidder better off without making some other bidder worse off
    - this definition can be modified to include reserve prices
  - not equivalent to efficiency: e.g., the greedy mechanism of [Lehmann, O'Callaghan and Shoham, 2002].
- ...maximizes an affine function
  - "affine maximizers": choose an allocation that maximizes $\sum_i \omega_i \widehat{v}_i(a_i) + \gamma_a$, given per-bidder $\omega$'s and per-allocation $\gamma$'s
  - maximal wrt $i$ as long as all $\gamma$ are finite, $\omega_i > 0$

[1] Our AAAI paper considers only this condition, and calls it maximality.

# Outline

# Single-minded bidders

In order to define a useful property, we must first define a special class of bidders.

**Definition (Single-minded Bidder)**

A bidder is single-minded if she has the valuation function:

$$\forall s \in 2^G, \qquad v_i(s) = \begin{cases} v_i > 0 & \text{if } s \supseteq b_i \\ 0 & \text{otherwise} \end{cases}$$

- The bundles $b_i$ are unknown to the auctioneer.
- We say that bidder $i$ wins if she is allocated at least $b_i$.
- Bidder $i$'s valuation for $b_i$ is denoted by $v_i$, and her declarations of this value and bundle are $\widehat{v}_i$ and $\widehat{b}_i$ respectively.

# Criticality

From necessary and sufficient conditions for DS truthful mechanisms (see e.g., [Bartal, Gonen & Nisan, 2003]) it can easily be shown that dominant-strategy truthful mechanisms offer critical values to single-minded bidders.

## Lemma (Criticality)

*If a deterministic, direct CA mechanism satisfies DS truthfulness, participation and consumer sovereignty, then for every bidder $i$, every $\widehat{v}_{-i}$ and every $s \in 2^G$, there exists a finite critical value $cv_i(s, \widehat{v}_{-i})$ where:*

- *if $\widehat{v}_i > cv_i(\widehat{b}_i)$, $i$ wins at least $\widehat{b}_i$ and pays $cv_i(\widehat{b}_i)$;*
- *if $\widehat{v}_i < cv_i(\widehat{b}_i)$, $i$ loses and pays 0.*

When $\widehat{v}_{-i}$ is understood from the context, we abbreviate $cv_i(s, \widehat{v}_{-i})$ as $cv_i(s)$.

# Outline

# Impossibility Result

## Theorem

*Let $M$ be a deterministic, direct CA mechanism that allows bidders to express single-minded preferences, and that satisfies*

- *DS truthfulness;*
- *participation;*
- *consumer sovereignty; and*
- *maximality with respect to at least 2 bidders $i$ and $j$.*

*Then $M$ is not revenue monotonic.*

# Proof sketch

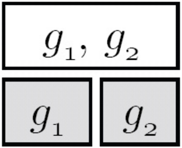
## Proof Sketch

*Consider three single-minded bidders.*

1. *Construct valuations by repeatedly probing the mechanism to determine the bidders' critical values given various declarations by the others.*
2. *Derive an expression for revenue with all three bidders.*
3. *Derive an expression for revenue without bidder* $1$.
4. *Show that (3) $>$ (2).*

# Proof – Constructing valuations

- Let $G = \{g_1, g_2\}$ and $N = \{1, 2, 3\}$
- Let $b_1 = \{g_1\}$, $b_2 = \{g_1, g_2\}$ and $b_3 = \{g_2\}$
- Let $i = 1$ and $j = 3$
- Define $v_1^* = \alpha_{1,b_1} + \varepsilon$ and $v_3^* = \alpha_{3,b_3} + \varepsilon$, for some $\varepsilon > 0$



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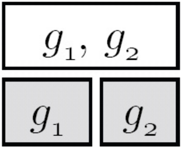


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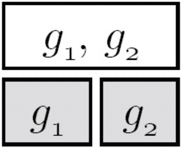


Select $v_1$, $v_2$ and $v_3$ as follows:

(by consumer sovereignty, these values are all finite)

1. $v_2 > cv_2(\emptyset, \emptyset, v_1^* + v_3^* + k)$, for some $k > 0$

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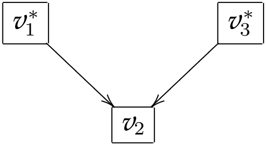

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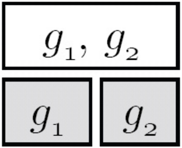
1. $v_2 > cv_2(\varnothing, \varnothing, v_1^* + v_3^* + k)$, for some $k > 0$
2. $v_1 > \max\{cv_1(\varnothing, v_2, v_3^*), cv_1(\varnothing, v_2, \varnothing), v_1^*\}$

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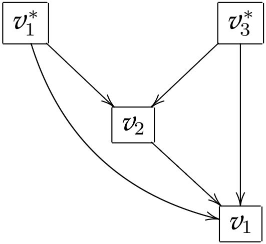

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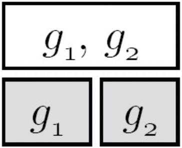


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1. $v_2 > cv_2(\varnothing, \varnothing, v_1^* + v_3^* + k)$, for some $k > 0$
2. $v_1 > \max\{cv_1(\varnothing, v_2, v_3^*), cv_1(\varnothing, v_2, \varnothing), v_1^*\}$
3. $v_3 > \max\{cv_3(v_1^*, v_2, \varnothing), cv_3(\varnothing, v_2, \varnothing), v_3^*\}$

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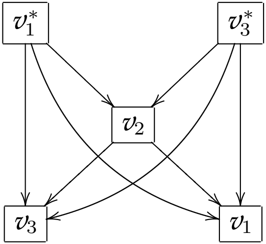

# Proof – Part 1: all Bidders Present

1. $v_1 > cv_1(\varnothing, v_2, v_3^*)$, so if bidder 3 bid $\langle b_3, v_3^* \rangle$ then
   - bidder 1 would win (by criticality lemma)
   - bidder 3 has the only non-overlapping bundle, and $v_3^* > \alpha_{3,b_3}$, so bidder 3 would also win (by maximality w.r.t. bidder 3)
   - therefore, $cv_3(v_1, v_2, \varnothing) \leq v_3^*$ (by criticality lemma)

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3. $v_1 > v_1^*$ and $v_3 > v_3^*$, so when all bidders bid $\langle b_i, v_i \rangle$:
   - bidders 1 and 3 win
   - bidder 2 pays zero (by participation)
   - the revenue of the auction is $R = cv_1(\varnothing, v_2, v_3) + cv_3(v_1, v_2, \varnothing)$ (by criticality lemma)

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**Conclusion (Part 1)**

$R \leq v_1^* + v_3^*$.

# Proof – Part 2: Bidder 1 not Present

1. $v_3 > cv_3(\varnothing, v_2, \varnothing)$, so
   - bidder $3$ wins (by criticality lemma)
   - $b_2$ and $b_3$ overlap so bidder $2$ cannot also win
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   - the revenue of the auction is $R_{-1} = cv_3(\varnothing, v_2, \varnothing)$

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2. $v_2 > cv_2(\varnothing, \varnothing, v_1^* + v_3^* + k)$, so
   - if bidder $3$ were to bid $\langle b_3, v_1^* + v_3^* + k \rangle$ then she would lose (by criticality lemma)
   - therefore $cv_3(\varnothing, v_2, \varnothing) \geq v_1^* + v_3^* + k$ (by criticality lemma)

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## Conclusion (Part 2)

$R_{-1} \geq v_1^* + v_3^* + k,\ k > 0.$

# Proof - Conclusion

## Conclusion (Part 1)

$R \leq v_1^* + v_3^*.$

## Conclusion (Part 2)

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## Conclusion (Overall)

$R < R_{-1}$; *therefore $M$ is not revenue monotonic.*

# Observations

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  - additional goods can be included in bundles or ignored

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  - instead, it makes requirements about how the mechanism behaves when bidders state single-minded preferences
- There need not be exactly three bidders or two goods
  - it's easy to construct valuations for additional bidders so that they play no role
  - additional goods can be included in bundles or ignored
- $R_{-1} - R = k$, a constant that we can set freely
  - thus the possible revenue gain from dropping one bidder is unbounded

# RM over the set of goods

## Corollary (RM over the set of goods)

*Let $M$ be a deterministic, direct CA mechanism that allows bidders to express single-minded preferences, and that satisfies*

- *DS truthfulness;*
- *participation;*
- *consumer sovereignty; and*
- *maximality with respect to at least 2 bidders $i$ and $j$.*

*Then $M$ is not revenue monotonic over the set of goods.*

- proof sketch: add an extra good to bidder 1's bundle and drop that good (which entails dropping bidder 1)
- note: works even without substitutes

# Pseudonomous bidding

## Corollary (pseudonymous bidding)

*Let $M$ be a deterministic, direct CA mechanism that allows bidders to express single-minded preferences, and that satisfies*

- *DS truthfulness;*
- *participation;*
- *consumer sovereignty; and*
- *maximality with respect to at least 2 bidders $i$ and $j$.*

*Then $M$ is not pseudonymous-bid proof.*

- proof sketch: in a world with only bidders 2 and 3, bidder 3 gains by pseudonymously bidding as bidder 1
- in previous literature (e.g., [Yokoo, 2006]) such a result is shown only for DS, efficient (i.e., Groves) mechanisms.

# Outline

# Conclusions

We have shown that reasonable, DS truthful CA mechanisms are not revenue monotonic.

Our result can be interpreted in several ways:

- in a DS mechanism, "don't leave money on the table" isn't an innocuous design decision.
- some "problems with VCG" are in fact properties of broad classes of CA mechanisms
- if you care about revenue in a CA, set reserve prices carefully and/or bundle goods
- "more competition" isn't the same as more bidders

# Future Work

Theoretical:

1. Look for necessary and/or sufficient conditions for revenue monotonicity

Experimental:

1. Conduct experiments to investigate the frequency and degree of RM failures in realistic settings
   - e.g., using test data from CATS
2. Find a DS truthful CA mechanism that violates RM with minimal probability