Polynomial Computation of Exact Correlated Equilibrium in Compact Games

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In a landmark paper, Papadimitriou and Roughgarden described a polynomial-time algorithm ("Ellipsoid Against Hope") for computing sample correlated equilibria of concisely-represented games. Recently, Stein, Parrilo and Ozdaglar showed that this algorithm can fail to find an exact correlated equilibrium, but can be easily modified to efficiently compute approximate correlated equilibria. It remained an open problem to determine whether the algorithm can be modified to compute an exact correlated equilibrium. In a new paper, we showed that it can, presenting a variant of the Ellipsoid Against Hope algorithm that guarantees the polynomial-time identification of exact correlated equilibrium. Our new algorithm differs from the original primarily in its use of a separation oracle that produces cuts corresponding to pure-strategy profiles. As a result, we no longer face the numerical precision issues encountered by the original approach, and both the resulting algorithm and its analysis are considerably simplified. Our new separation oracle can be understood as a derandomization of Papadimitriou and Roughgarden's original separation oracle via the method of conditional expectations.

Categories and Subject Descriptors: F.2.0 [**Analysis of Algorithms and Problem Complexity**]: General

General Terms: Algorithms, Economics, Theory

Additional Key Words and Phrases: correlated equilibrium, ellipsoid method, separation oracle, derandomization

Consider the problem of computing a sample correlated equilibrium (CE) given a finite, simultaneous-move game. It is known that correlated equilibria of a game can be formulated as probability distributions over pure strategy profiles satisfying certain linear constraints. The resulting linear feasibility program has size polynomial in the size of the normal form representation of the game. However, the size of the normal form representation grows exponentially in the number of players. This is problematic when games involve large numbers of players. Fortunately, most large games of practical interest have highly-structured payoff functions. A line of research thus exists to look for *compact game representations* that are able to succinctly describe structured games. But now the size of the linear feasibility program for CE can be exponential in the size of representation; furthermore a CE can require exponential space to specify.

The "Ellipsoid Against Hope" algorithm [Papadimitriou 2005; Papadimitriou and Roughgarden 2008] is a polynomial-time method for identifying a (polynomial-size representation of a) CE, given a game representation satisfying two properties: *polynomial type*, which requires that the number of players and the number of actions for each player are bounded by polynomials in the size of the representation, and the *polynomial expectation property*, which requires access to a polynomial-

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time algorithm that computes the expected utility of any player under any product distribution. Many existing compact game representations (including graphical games, symmetric games, congestion games, polymatrix games and action-graph games) satisfy these properties.

At a high level, the Ellipsoid Against Hope algorithm works by solving an infeasible dual LP using the ellipsoid method (exploiting the existence of a separation oracle), and arguing that the LP formed by the generated cutting planes must also be infeasible. Solving the dual of this latter LP (which has polynomial size) yields a CE, which is represented as a mixture of the product distributions generated by the separation oracle.

In a recent paper, Stein et al. [2010] raised two interrelated concerns about the Ellipsoid Against Hope algorithm. First, they identified a symmetric 3-player, 2-action game with rational utilities on which the algorithm can fail to compute an exact CE. Second, they also showed that the original analysis in [Papadimitriou and Roughgarden 2008] incorrectly handles certain numerical precision issues, which we now briefly describe. Recall that a run of the ellipsoid method requires as inputs an initial bounding ball with radius R and a volume bound v such that the algorithm stops when the ellipsoid's volume is smaller than v. To correctly certify the (in)feasibility of an LP using the ellipsoid method, R and v need to be set to appropriate values, which depend on the maximum encoding size of a constraint in the LP. However (as pointed out by Papadimitriou and Roughgarden [2008]), each cut returned by the separation oracle is a convex combination of the constraints of the original dual LP and thus may require more bits to represent than any of the constraints; as a result, the infeasibility of the LP formed by these cuts is not guaranteed. Papadimitriou and Roughgarden [2008] proposed a method to overcome this difficulty, but Stein et al. [2010] showed that this method is insufficient for finding an exact CE. For the related problem of finding an approximate correlated equilibrium (ϵ-CE), Stein et al. [2010] gave a slightly modified version of the Ellipsoid Against Hope algorithm that runs in time polynomial in $\log \frac{1}{\epsilon}$ and the game representation size. For problems that can have necessarily irrational solutions, it is standard to consider such approximations as efficient; however, there always exists a rational CE in a game with rational utilities, since CE are defined by linear constraints. It remained an open problem to determine whether the Ellipsoid Against Hope algorithm can be modified to compute an exact, rational correlated equilibrium. We refer interested readers to [Papadimitriou and Roughgarden 2010] for recent discussions of these issues. We note also that Stein et al. have recently withdrawn their paper from arXiv. It is our impression that their results are nevertheless still believed to be correct.

In a new paper [Jiang and Leyton-Brown 2010], we used an alternate approach—completely sidestepping the issues just discussed—to derive a polynomial-time algorithm for computing an exact (and rational) correlated equilibrium given a game representation that has polynomial type and satisfies the polynomial expectation property. Our approach is based on the observation that if we use a separation oracle (for the same dual LP formulation as in [Papadimitriou and Roughgarden 2008]) that generates cuts corresponding to pure-strategy profiles (instead of Papadimitriou and Roughgarden [2008]'s separation oracle that generates nontrivial product distributions), then these cuts are actual constraints in the dual LP, as opposed to the convex combinations of constraints produced by Papadimitriou and Roughgarden [2008]'s separation oracle. As a result we no longer encounter the numerical accuracy issues that prevented the previous approaches from finding exact correlated equilibria. Both the resulting algorithm and its analysis are also considerably simpler than the original: standard techniques from the theory of the ellipsoid method are sufficient to show that our algorithm computes an exact CE using a polynomial number of oracle queries.

The key issue is the identification of pure-strategy-profile cuts. It is relatively straightforward to show that such cuts always exist: since the product distribution generated by the Ellipsoid Against Hope algorithm ensures the nonnegativity of a certain expected value, then by a simple application of the probabilistic method there must exist a pure-strategy profile that also ensures the nonnegativity of that expected value. The key is to go beyond this nonconstructive proof of existence to also *compute* pure-strategy-profile cuts in polynomial time. We showed how to do this by applying the method of conditional expectations, an approach for derandomizing probabilistic proofs of existence. At a high level, our new separation oracle begins with the product distribution generated by Papadimitriou and Roughgarden [2008]'s separation oracle, then sequentially fixes a pure strategy for each player in a way that guarantees that the corresponding conditional expectation given the choices so far remains nonnegative.

Another effect of our use of pure-strategy-profile cuts is that the correlated equilibria generated by our algorithm are guaranteed to have polynomial-sized supports; i.e., they are mixtures over a polynomial number of pure strategy profiles. Correlated equilibria with polynomial-sized supports are known to exist in every game (e.g., [Germano and Lugosi 2007]) but no tractable algorithm has previously been proposed for identifying them. Such small-support correlated equilibria have a simpler form than the mixtures of product distributions produced by the Ellipsoid Against Hope algorithm.

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