Reasoning Under Uncertainty: Variable Elimination

CPSC 322 Lecture 29

Learning Goals for today's class

You can:

- Carry out **variable elimination** by using factor representation and using the factor operations.
- Use techniques to simplify variable elimination.

Lecture Overview

- **Recap Intro Variable Elimination**
- Variable Elimination
  - Simplifications
  - Example
  - Independence
- Where are we?

Bnet Inference: General

- Suppose the variables of the belief network are $X_1, \ldots, X_n$.
- Z is the query variable
- $Y_1 = v_1, \ldots, Y_j = v_j$ *are the observed variables (with their values)*
- $Z_1, \ldots, Z_k$ are the remaining variables

- What we want to compute: $P(Z \mid Y_1 = v_1 \ldots, Y_j = v_j)$

- We can actually compute: $P(Z, Y_1 = v_1 \ldots, Y_j = v_j)$

$$P(Z \mid Y_1 = v_1 \ldots, Y_j = v_j) = \frac{P(Z, Y_1 = v_1 \ldots, Y_j = v_j)}{P(Y_1 = v_1 \ldots, Y_j = v_j)} = \frac{P(Z, Y_1 = v_1 \ldots, Y_j = v_j)}{\sum_Z P(Z, Y_1 = v_1, \ldots Y_j = v_j)}$$

product rule

treat denominator as a constant, so normalize at the end instead

Inference with Factors

We can compute $P(Z, Y_1=v_1, \ldots, Y_j=v_j)$ by

- expressing the joint probability distribution (JPD) as a factor

$$f(Z, Y_1 \ldots, Y_j, Z_1 \ldots, Z_j)$$

- **assigning** $Y_1=v_1, \ldots, Y_j=v_j$
- **summing out** the variables $Z_1, \ldots, Z_k$

But the whole point of BNets was to get rid of the JPD

Variable Elimination Intro (1)

$$P(Z, Y_1 = v_1, \ldots, Y_j = v_j) = \sum_{Z_k} \ldots \sum_{Z_1} f(Z, Y_1, .., Y_j, Z_1, .., Z_k)_{Y_1 = v_1 \ \ldots \ Y_j = v_j}$$

- Using the **chain rule** and the **definition of a BNet**, we can write $P(X_1, \ldots, X_n)$ as $\prod_{i=1}^{n} P(X_i \mid pX_i)$ ← **parents of X_i**

- We can express the joint factor as a product of factors

$f(Z, Y_1 \ldots, Y_j, Z_1 \ldots, Z_j)$ $\prod_{i=1}^{n} f(X_i, pX_i)$

$$P(Z, Y_1 = v_1, \ldots, Y_j = v_j) = \sum_{Z_k} \ldots \sum_{Z_1} \prod_{i=1}^{n} f(X_i, pX_i)_{Y_1 = v_1 \ \ldots \ Y_j = v_j}$$

Variable Elimination Intro (2)

Inference in belief networks thus reduces to computing **sums of products**….

1. Construct a factor for each conditional probability.
2. In each factor assign the observed variables to their observed values.
3. Multiply the factors
4. For each of the other variables $Z_i \in \{Z_1, \ldots, Z_k\}$, sum out Z_i

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How to simplify the Computation?

- Assume we have turned the CPTs into factors and performed the assignments

Let's focus on the basic case, for instance…

$$\sum_{A} f(C,D) \times f(A,B,D) \times f(E,A) \times f(D)$$

How to simplify: basic case

Let's focus on the basic case. $\sum_{Z_1} \prod_{i=1}^{n} f(\text{varsX}_i)$

$$\sum_{A} f(C,D) \times f(A,B,D) \times f(E,A) \times f(D)$$

How can we compute efficiently?

Factor out those terms that don't involve Z_1 !

$$\left(\prod_{i | Z_1 \notin \text{varsXi}} f(\text{varsX}_i) \right) \times \left(\sum_{Z_1} \prod_{i | Z_1 \in \text{varsXi}} f(\text{varsX}_i) \right)$$

$$f(C,D) \times f(D) \times \sum_{A} f(A,B,D) \times f(E,A)$$

Analogy with "Computing sums of products"

This simplification is similar to what you can do in basic algebra with *multiplication* and *addition*

- It takes *14* multiplications or additions to evaluate the expression

$$a\,b + a\,c + a\,d + a\,e\,h + a\,f\,h + a\,g\,h$$

- This expression be evaluated more efficiently (only 7 operations)….

$$a * (b + c + d + h * (e + f + g))$$

Variable elimination ordering

Is there only one way to simplify? ***NO***

$$P(G,D{=}t) = \sum_{A,B,C,} f(A,G)\, f(B,A)\, f(C,G)\, f(B,C)$$

$$P(G,D{=}t) = \sum_{A} f(A,G) \sum_{B} f(B,A) \sum_{C} f(C,G)\, f(B,C)$$

$$P(G,D{=}t) = \sum_{A} f(A,G) \sum_{C} f(C,G) \sum_{B} f(B,C)\, f(B,A)$$

Variable elimination algorithm: Summary

$$P(Z, Y_1 \ldots, Y_j, Z_1 \ldots, Z_j)$$

To compute $P(Z| Y_1=v_1, \ldots, Y_j=v_j)$:

1. Construct a factor for each conditional probability.
2. Set the observed variables to their observed values.
3. Given an elimination ordering, simplify/decompose sum of products
4. Perform products and sum out Z_i
5. Multiply the remaining factors (all in ? **A.** $Y_1=v_1$ **B.** Y_2 **C.** Z_2 **D.** Z)
6. Normalize: divide the resulting factor $f(Z)$ by $\sum_Z f(Z)$.

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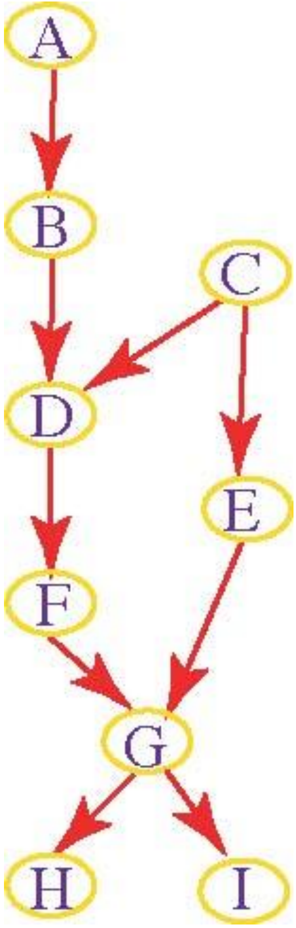
Variable elimination example

Compute $P(G \mid H=h_1)$.

- $P(G,H) = \sum_{A,B,C,D,E,F,I} P(A,B,C,D,E,F,G,H,I)$



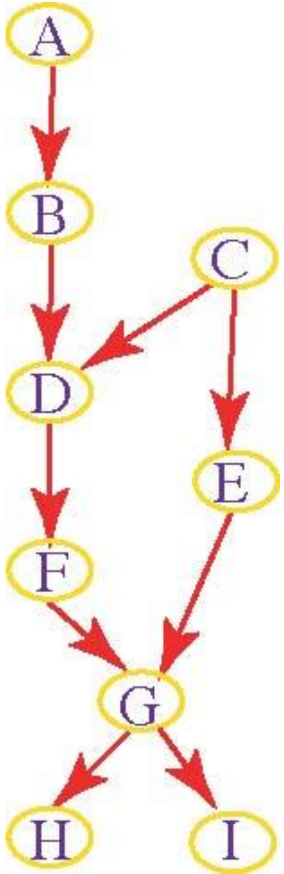
Variable elimination example

Compute $P(G \mid H=h_1)$.

- $P(G,H) = \sum_{A,B,C,D,E,F,I} P(A,B,C,D,E,F,G,H,I)$

Chain Rule + Conditional Independence:

$$P(G,H) = \sum_{A,B,C,D,E,F,I} P(A)P(B|A)P(C)P(D|B,C)P(E|C)P(F|D)P(G|F,E)P(H|G)P(I|G)$$



Variable elimination example (step1)

Compute $P(G \mid H=h_1)$.

- $P(G,H) = \sum_{A,B,C,D,E,F,I} P(A)P(B|A)P(C)P(D|B,C)P(E|C)P(F|D)P(G|F,E)P(H|G)P(I|G)$

Factorized Representation:

$$P(G,H) = \sum_{A,B,C,D,E,F,I} f_0(A)\ f_1(B,A)\ f_2(C)\ f_3(D,B,C)\ f_4(E,C)\ f_5(F, D)\ f_6(G,F,E)\ f_7(H,G)\ f_8(I,G)$$

- $f_0(A)$
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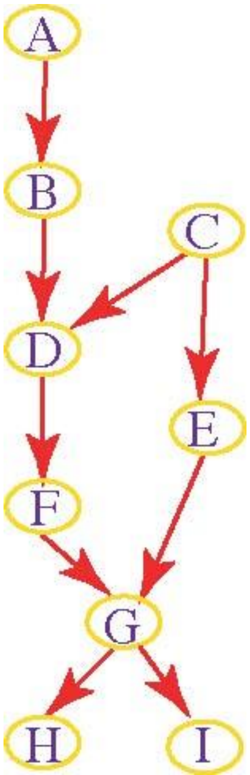


Variable elimination example (step 2)

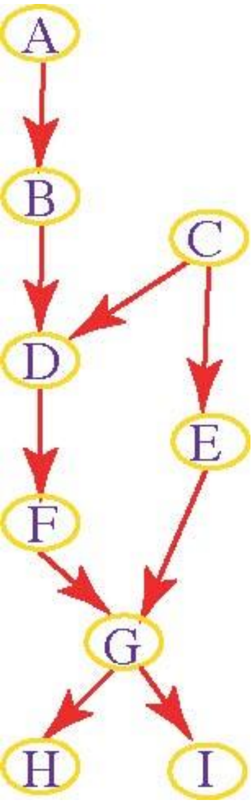
Compute $P(G \mid H=h_1)$.

Previous state:

$$P(G,H) = \sum_{A,B,C,D,E,F,I} f_0(A)\, f_1(B,A)\, f_2(C)\, f_3(D,B,C)\, f_4(E,C)\, f_5(F, D)\, f_6(G,F,E)\, f_7(H,G)\, f_8(I,G)$$

Observe H :

$$P(G,H=h_1) = \sum_{A,B,C,D,E,F,I} f_0(A)\, f_1(B,A)\, f_2(C)\, f_3(D,B,C)\, f_4(E,C)\, f_5(F, D)\, f_6(G,F,E)\, f_9(G)\, f_8(I,G)$$



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- $f_7(H,G)$
- $f_8(I,G)$

- $f_9(G)$

Variable elimination example (steps 3-4)

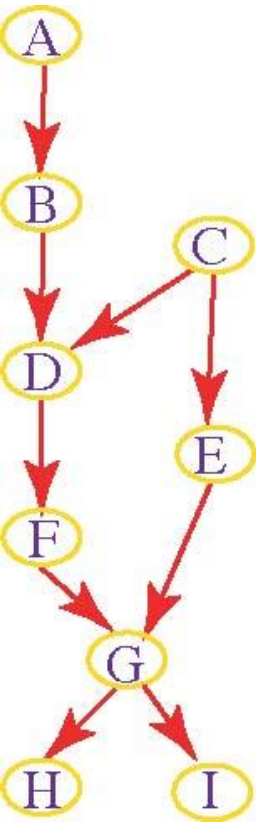
Compute $P(G \mid H=h_1)$.

Previous state:

$$P(G,H) = \sum_{A,B,C,D,E,F,I} f_0(A)\, f_1(B,A)\, f_2(C)\, f_3(D,B,C)\, f_4(E,C)\, f_5(F, D)\, f_6(G,F,E)\, f_9(G)\, f_8(I,G)$$

Elimination ordering *A, C, E, I, B, D, F* :

$$P(G,H=h_1) = f_9(G) \sum_F \sum_D f_5(F, D) \sum_B \sum_I f_8(I,G) \sum_E f_6(G,F,E) \sum_C f_2(C)\, f_3(D,B,C)\, f_4(E,C) \sum_A f_0(A)\, f_1(B,A)$$



- $f_0(A)$
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- $f_7(H,G)$
- $f_8(I,G)$

- $f_9(G)$

Variable elimination example(steps 3-4)

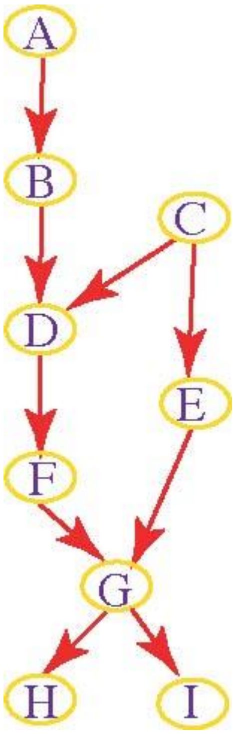
Compute $P(G \mid H=h_1)$. Elimination ordering *A, C, E, I, B, D, F.*

Previous state:

$P(G,H=h_1) = f_9(G) \sum_F \sum_D f_5(F, D) \sum_B \sum_I f_8(I,G) \sum_E f_6(G,F,E) \sum_C f_2(C)\, f_3(D,B,C)\, f_4(E,C) \sum_A f_0(A)\, f_1(B,A)$

Eliminate A:

$P(G,H=h_1) = f_9(G) \sum_F \sum_D f_5(F, D) \sum_B f_{10}(B) \sum_I f_8(I,G) \sum_E f_6(G,F,E) \sum_C f_2(C)\, f_3(D,B,C)\, f_4(E,C)$



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- $f_7(H,G)$
- $f_8(I,G)$
- $f_9(G)$
- $f_{10}(B)$

Variable elimination example(steps 3-4)

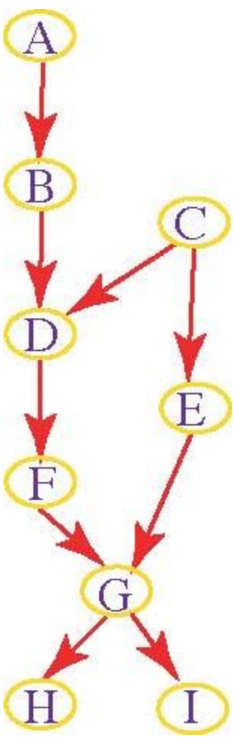
Compute $P(G \mid H=h_1)$. Elimination ordering *A, C, E, I, B, D, F.*

Previous state:

$P(G,H=h_1) = f_9(G) \sum_F \sum_D f_5(F, D) \sum_B f_{10}(B) \sum_I f_8(I,G) \sum_E f_6(G,F,E) \sum_C f_2(C)\, f_3(D,B,C)\, f_4(E,C)$

Eliminate C:

$P(G,H=h_1) = f_9(G) \sum_F \sum_D f_5(F, D) \sum_B f_{10}(B) \sum_I f_8(I,G) \sum_E f_6(G,F,E)\, f_{12}(B,D,E)$



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- $f_8(I,G)$

- $f_9(G)$
- $f_{10}(B)$
- $f_{12}(B,D,E)$

Variable elimination example(steps 3-4)

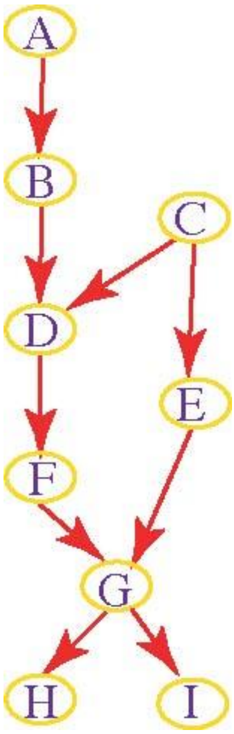
Compute $P(G \mid H=h_1)$. Elimination ordering A, C, E, I, B, D, F.

Previous state:

$P(G,H=h_1) = f_9(G) \sum_F \sum_D f_5(F, D) \sum_B f_{10}(B) \sum_I f_8(I,G) \sum_E f_6(G,F,E)\, f_{12}(B,D,E)$

Eliminate E:

$P(G,H=h_1) = f_9(G) \sum_F \sum_D f_5(F, D) \sum_B f_{10}(B)\, f_{13}(B,D,F,G) \sum_I f_8(I,G)$



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- $f_{10}(B)$
- $f_{12}(B,D,E)$
- $f_{13}(B,D,F,G)$

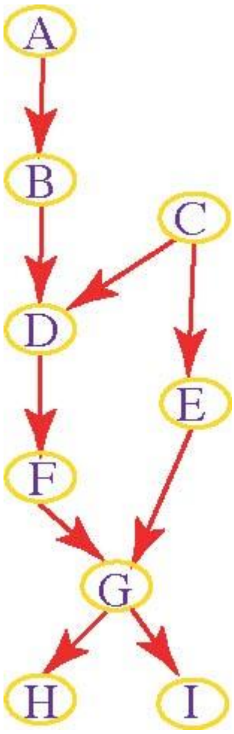
Variable elimination example(steps 3-4)

Compute $P(G \mid H=h_1)$. Elimination ordering *A, C, E, I, B, D, F.*

Previous state: $P(G,H=h_1) = f_9(G) \sum_F \sum_D f_5(F, D) \sum_B f_{10}(B)\, f_{13}(B,D,F,G) \sum_I f_8(I,G)$

Eliminate I:

$$P(G,H=h_1) = f_9(G)\, f_{14}(G) \sum_F \sum_D f_5(F, D) \sum_B f_{10}(B)\, f_{13}(B,D,F,G)$$



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- $f_{10}(B)$
- $f_{12}(B,D,E)$
- $f_{13}(B,D,F,G)$
- $f_{14}(G)$

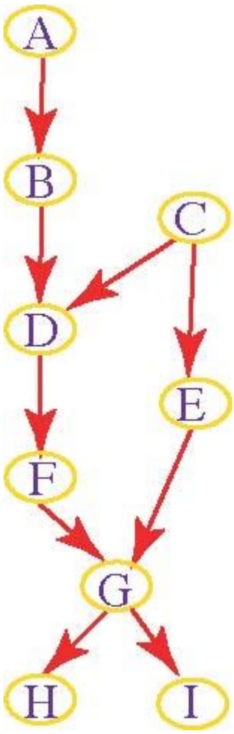
Variable elimination example(steps 3-4)

Compute $P(G \mid H=h_1)$. Elimination ordering *A, C, E, I, B, D, F.*

Previous state: $P(G,H=h_1) = f_9(G)\, f_{14}(G) \sum_F \sum_D f_5(F, D) \sum_B f_{10}(B)\, f_{13}(B,D,F,G)$

Eliminate B:

$P(G,H=h_1) = f_9(G)\, f_{14}(G) \sum_F \sum_D f_5(F, D)\, f_{15}(D,F,G)$



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- $f_{13}(B,D,F,G)$
- $f_{14}(G)$
- $f_{15}(D,F,G)$

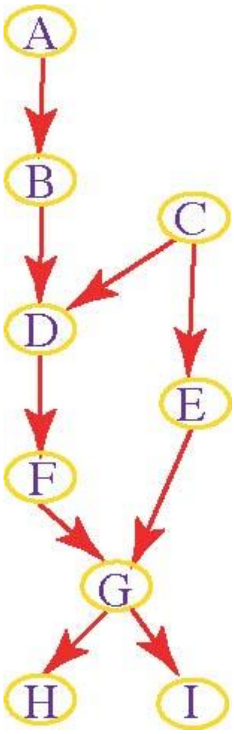
Variable elimination example(steps 3-4)

Compute $P(G \mid H=h_1)$. Elimination ordering A, C, E, I, B, D, F.

Previous state: $P(G,H=h_1) = f_9(G)\, f_{14}(G) \sum_F \sum_D f_5(F, D)\, f_{15}(D,F,G)$

Eliminate D:

$P(G,H=h_1) = f_9(G)\, f_{14}(G) \sum_F f_{16}(F, G)$



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- $f_{15}(D,F,G)$
- $f_{16}(F, G)$

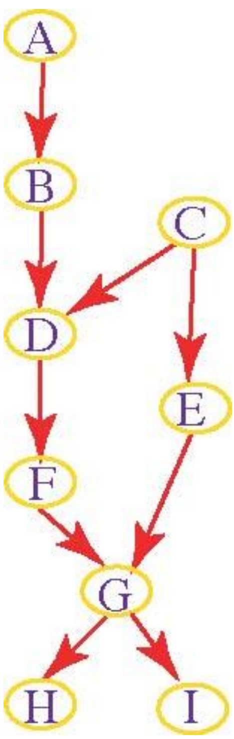
Variable elimination example(steps 3-4)

Compute $P(G \mid H=h_1)$. Elimination ordering *A, C, E, I, B, D, F.*

Previous state: $P(G,H=h_1) = f_9(G)\, f_{14}(G) \sum_F f_{16}(F, G)$

Eliminate F:

$P(G,H=h_1) = f_9(G)\, f_{14}(G)\ f_{17}(G)$



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- $f_{17}(G)$

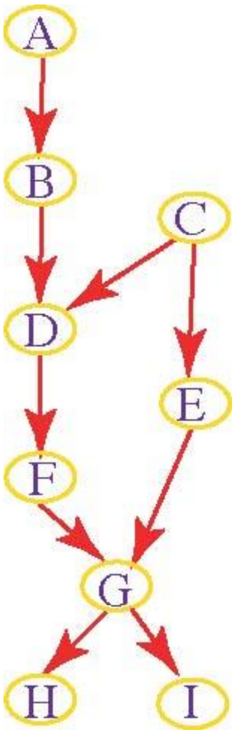
Variable elimination example (step 5)

Compute $P(G \mid H=h_1)$. Elimination ordering *A, C, E, I, B, D, F.*

Previous state: $P(G,H=h_1) = f_9(G)\, f_{14}(G)\, f_{17}(G)$

Multiply remaining factors:

$P(G,H=h_1) = f_{18}(G)$



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- $f_{16}(F, G)$
- $f_{17}(G)$
- $f_{18}(G)$

Variable elimination example (step 6)

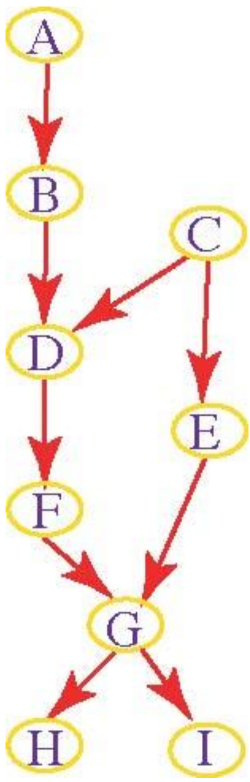
Compute $P(G \mid H=h_1)$. Elimination ordering *A, C, E, I, B, D, F.*

Previous state:

$P(G,H=h_1) = f_{18}(G)$

Normalize:

$$P(G \mid H=h_1) = f_{18}(G) / \sum_{g \in dom(G)} f_{18}(G)$$



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  - Example
  - Independence
- Where are we?

Complexity (not required)

- The complexity of the algorithm depends on a measure of complexity of the network.
- The size of a tabular representation of a factor is exponential in the number of variables in the factor.
- The **treewidth** of a network, given an elimination ordering, is the maximum number of variables in a factor created by summing out a variable, given the elimination ordering.
- The **treewidth** of a belief network is the minimum treewidth over all elimination orderings. The treewidth depends only on the graph structure and is a measure of the sparseness of the graph.
- The complexity of VE is exponential in the treewidth and linear in the number of variables.
- Finding the elimination ordering with minimum treewidth is NP-hard, but there is some good elimination ordering heuristics.

Variable elimination and conditional independence

- Variable Elimination looks incredibly painful for large graphs
- We used conditional independence…..

$$\prod_{i=1}^{n} P(X_i \mid pX_i)$$

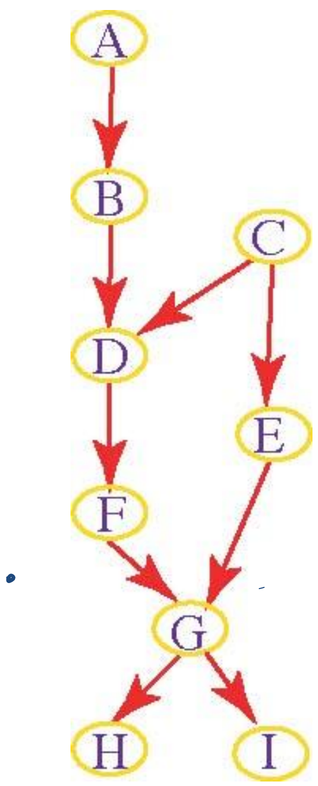
  - Can we use it to make variable elimination simpler?

Yes, all the variables from which the query is conditionally independent given the observations can be pruned from the BNet

Unobserved leaf nodes can also be pruned

VE and conditional independence: Example

All the variables from which the query is conditionally independent given the observations (and unobserved leaf nodes) can be pruned from the BNet



e.g., P(G | H= v_1***, F=*** v_2***, C=*** v_3***).***

A. ***B, D, E***

B. ***E, D***

C . ***A, I***

D. ***B, D, A, I***

E. ***E, I, E, I, O***

Next Class

Probability and Time *(Textbook 8.5)*