Bottom-Up: Soundness and Completeness

CPSC 322 Lecture 20

Lecture Overview

- Recap
- Soundness and Completeness
- Soundness of Bottom-up Proofs
- Completeness of Bottom-up Proofs

(Propositional) Logic: Key ideas

Given a domain that can be represented with **n propositions** you have ____________ interpretations (possible worlds)

If you do not know anything, then you can be in any of those possible worlds

If you know that some **facts** and some **logical formulas** are true (i.e. you have a _______________), then you know that you can be only in interpretations that _______________________

It would be nice to know what else is true (i.e. what is ____________ by the knowledge base)

PDCL syntax / semantics / proofs

Consider a domain represented by three propositions: *p, q, r*

$$KB = \begin{cases} q. \\ r. \\ p \leftarrow q \wedge r. \end{cases}$$

What are the models?

What is logically entailed ?

Interpretations

r	*q*	*p*
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

Prove $G = (q \wedge p)$ is entailed by the KB

PDCL syntax / semantics / proofs

$$KB = \begin{cases} p \leftarrow q \wedge r. \\ q. \end{cases}$$

Interpretations

r	*q*	*p*
T	T	T
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What are the models?

What is logically entailed ?

Prove $G = (q \wedge p)$ is entailed by the KB? **We can't**

The Bottom-Up Proof Procedure

Based on generalized **modus ponens**

$KB \vdash G$ if $G \subseteq C$ at the end of this procedure:

C :={};

repeat

- **select** clause "$h \leftarrow b_1 \wedge \ldots \wedge b_m$" in KB such that $b_i \in C$ for all i, and $h \notin C$;
- $C := C \cup \{ h \};$

until no more clauses can be selected.

Learning Goals for this class

You can:

- Prove that the Bottom-Up proof procedure is **sound**
- Prove that the Bottom-Up proof procedure is **complete**

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Soundness and Completeness

Generic Soundness of proof procedure:
If G can be proved by the procedure (KB $\vdash$ G)
then G is logically entailed by the KB (KB $\models$ G)

Generic Completeness of proof procedure:
If G is logically entailed by the KB (KB $\models$ G)
then G can be proved by the procedure (KB $\vdash$ G)

In other words:

- Everything **derived** from a **sound** proof procedure is **entailed** by the KB
- Everything **entailed** by the KB can be **derived** from a **complete** proof procedure

An exercise for you

Compare two "proof procedures" for PDCL

X. C_X = {All clauses in KB with empty bodies}

Y. C_Y = {All atoms in KB}

KB

$a \leftarrow e \wedge g.$

$b \leftarrow f \wedge g.$

$c \leftarrow e.$

$f \leftarrow c$

$e.$

$d.$

iclicker

A. Both X and Y are sound and complete

B. Both X and Y are neither sound nor complete

C. X is sound only and Y is complete only

D. X is complete only and Y is sound only

E. I only care if it's on the final exam

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Soundness of bottom-up proof procedure

$KB \vdash G$ if $G \subseteq C$ at the end of this procedure:

C :={};
repeat

- **select** clause "$h \leftarrow b_1 \wedge \ldots \wedge b_m$" in KB such that $b_i \in C$ for all i, and $h \notin C$;
- $C := C \cup \{ h \};$

until no more clauses can be selected.

So BU is sound, if all the atoms in C are logically entailed by the KB

Soundness of bottom-up proof procedure

Suppose this is not the case.

1. Let *h* be the first atom added to *C* that is not **entailed** by KB (i.e., that's not **true in every model of *KB***)

2. Suppose *h* isn't true in model ***M*** of *KB*.

3. Since *h* was added to C, there must be a clause in KB of the form: $h \leftarrow b_1 \wedge \ldots \wedge b_m$

4. Each b_i is true in *M (because of 1.)*. *h* is false in *M.* So the clause is false in *M*.

5. Therefore ***M is not a model***

6. **Contradiction!** thus no such *h* exists.

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Completeness of Bottom Up

Generic Completeness of proof procedure:

If G is logically entailed by the KB (KB ⊧ G)

then G can be proved by the procedure (KB ⊢ G)

Sketch of our proof:

1. Suppose *KB* ⊧ *G*. Then G is true in all models of *KB*.
2. Thus *G* is true in any particular model of KB.
3. We will define a particular model such that if G is true in that model, G is proved by the bottom up algorithm.
4. Thus $KB \vdash_{BU} G$.

Let’s work on step 3

3. We will define a model such that if G is true in that model, G is proved by the bottom up algorithm.

3.1 We will define a particular interpretation $\mathcal{I}$ such that iff G is true in $\mathcal{I}$, G is proved by the bottom up algorithm.

3.2 We will then show that $\mathcal{I}$ is a model

Let's work on step 3.1

3.1 Define interpretation $\mathcal{I}$ so that if G is true in $\mathcal{I}$,

Then $G \subseteq C$.

Let I be the interpretation in which every element of C is **true** and every other atom is **false**.

KB

$a \leftarrow e \wedge g.$

$b \leftarrow f \wedge g.$

$c \leftarrow e.$

$f \leftarrow c \wedge e.$

$e.$

$d.$

C= { *c*, *d*, *e*, *f* }

{ *a*,	*b*,	*c*,	*d*,	*e*,	*f*,	*g* }
F	*F*	*T*	*T*	*T*	*T*	*F*

Let's work on step 3.2

Claim: I is a model of *KB* *(we'll call it the **minimal model**)*.

Proof: **Assume** that I is not a model of *KB*.

- **Then** there must exist some clause $h \leftarrow b_1 \wedge \ldots \wedge b_m$ in *KB* (having zero or more b_i's) which is **false** in I.
- The only way this can occur is if $b_1 \ldots b_m$ are **true** in I (i.e., are in C) and h is **false** in I (i.e., is not in C)
- **But** if each b_i belonged to C, Bottom Up would have added h to C as well.
- So, there can be no clause in the KB that is false in interpretation I (which implies the claim :-)

Completeness of Bottom Up
(proof summary)

If *KB* ⊧ *G* then KB $\vdash_{BU}$ *G*

- Suppose *KB* ⊧ *G* .
- Then *G* is true in all models of KB
- Thus *G* is true in the minimal model
- Thus G ⊆ C
- Thus *G* is proved by BU
- i.e., KB $\vdash_{BU}$ *G*

Next class

- Using PDC Logic to model the electrical domain
- Reasoning in the electrical domain
- Top-down proof procedure (as Search!)
- Datalog