Linear Convergence under the Polyak-Łojasiewicz Inequality

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Linear Convergence of Gradient-Based Methods

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⋆ **This talk**: How much can we relax strong-convexity?

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Polyak-Łojasiewicz Inequality

- Polyak [1963] showed linear convergence of GD assuming

$$\frac{1}{2}\|\nabla f(x)\|^2 \geq \mu\left(f(x) - f^*\right),$$

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- A special case of Łojasiewicz' inequality [1963].
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- Using the PL inequality, we show

Smoothness + **PL Inequality** ~~Strong-Convexity~~ ⇒ Linear Convergence

- Consider the basic unconstrained smooth optimization problem,

$$\min_{x \in \mathbb{R}^d} \; f(x),$$

where f satisfies the PL inequality and ∇f is Lipschitz continuous,

$$f(y) \le f(x) + \langle \nabla f(x), y - x \rangle + \frac{L}{2}\|y - x\|^2.$$

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- Subtracting f^* and applying recursively gives global linear rate,

$$f(x^k) - f^* \leq \left(1 - \frac{\mu}{L}\right)^k \left[f(x^0) - f^*\right].$$

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- Does not imply convexity (unlike strong-convexity).

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  - WSC: weak strong convexity [Necoara et al., 2015].
    - Also sometimes used for QG $+$ convexity.
- Proofs are more complicated under these conditions.
- Are they more general?

Relationships Between Conditions

Theorem

For a function f with a Lipschitz-continuous gradient, we have:

$$\text{(SC)} \rightarrow \text{(ESC)} \rightarrow \text{(WSC)} \rightarrow \text{(RSI)} \rightarrow \text{(EB)} \equiv \text{(PL)} \rightarrow \text{(QG)}.$$

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- QG is the weakest condition but allows non-global local minima.
- PL $\equiv$ EB are most general conditions.
  - Allow linear convergence to global minimizer.

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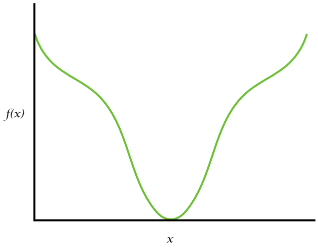
  - Example of invex but non-convex function satisfying PL:

$$f(x) = x^2 + 3\sin^2(x)$$



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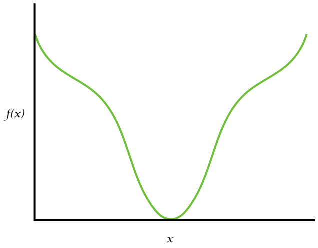

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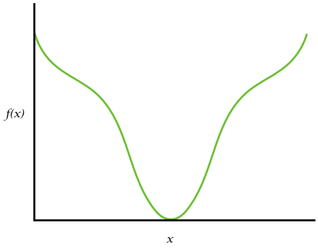

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- For these problems we often divide analysis into two phases:
  - Global convergence: iterations needed to get "close" to minimizer.
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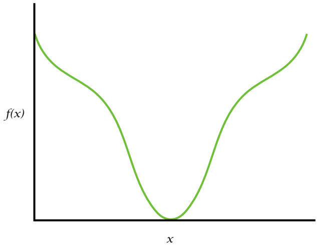

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  - Global convergence: iterations needed to get "close" to minimizer.
  - Local convergence: how fast does it converge near the minimizer?
- Usually, local convergence assumes strong-convexity near minimizer.
  - If we assume PL, then local convergence phase may be much earlier.

Convergence for Huge-Scale Methods

- For large datasets, we typically don't use GD.
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  - But the PL inequality can be used to analyze other algorithms.
- We will use PL for coordinate descent and stochastic gradient.
  - Garber & Hazan [2015] consider Franke-Wolfe.
  - Reddi et al. [2016] consider other stochastic algorithms.
  - In Karimi et al. [2016], we consider sign-based gradient methods.

Random and Greedy Coordinate Descent

- For randomized coordinate descent under PL we have

$$\mathbb{E}\left[f(x^k) - f^*\right] \leq \left(1 - \frac{\mu}{dL_c}\right)^k \left[f(x^0) - f^*\right],$$

  where L_c is coordinate-wise Lipschitz constant of ∇f.

- Faster than GD rate if iterations are d times cheaper.

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- Gives rate for some boosting variants [Meir and Rätsch, 2003].

- Stochastic gradient (SG) methods apply to general problems

$$\operatorname*{argmin}_{x \in R^d} f(x) = \mathbb{E}[f_i(x)],$$

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- SG methods use the iteration

$$x^{k+1} = x^k - \alpha_k \nabla f_{i_k}(x^k),$$

where ∇f_{i_k} is an unbiased gradient approximation.

Theorem

With $\alpha_k = \frac{2k+1}{2\mu(k+1)^2}$ *the SG method satisfies*

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while with α_k set to constant α we have

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- Fast reduction of sub-optimality under small constant step-size.

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- $O(1/k)$ rate without strong-convexity (or even convexity).
- Fast reduction of sub-optimality under small constant step-size.
- Our work and Reddi et al. [2016] consider finite sum case:
    - Analyze stochastic variance-reduced gradient (SVRG) method.
    - Obtain linear convergence rates.

PL Generalizations for Non-Smooth Problems

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- But proximal-gradient methods are more relevant for ML.
  - KL inequality has been used to show local rate for this method.
- We propose a different PL generalization giving a simple global rate.

Proximal-PL Inequality

- Proximal-gradient methods apply to the problem

$$\underset{x \in \mathbb{R}^d}{\text{argmin}}\ F(x) = f(x) + g(x),$$

where ∇f is L-Lipschitz and g is a potentially non-smooth convex function.

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- Condition yields extremely-simple proof:

$$\begin{aligned}
F(x^{k+1}) &= f(x^{k+1}) + g(x^k) + g(x^{k+1}) - g(x^k) \\
&\leq F(x^k) + \langle \nabla f(x^k), x^{k+1} - x^k \rangle + \frac{L}{2}\|x^{k+1} - x^k\|^2 + g(x^{k+1}) - g(x^k) \\
&\leq F(x^k) - \frac{1}{2L}\mathcal{D}_g(x^k, L) \\
&\leq F(x^k) - \frac{\mu}{L}\left[F(x^k) - F^*\right] \Rightarrow F(x^k) - F^* \leq \left(1 - \frac{\mu}{L}\right)^k \left[F(x^0) - F^*\right]
\end{aligned}$$

- We also analyze proximal coordinate descent under PL.
  - Reddi et al. [2016] analyze proximal-SVRG and proximal-SAGA.

Relevant Problems for Proximal-PL

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- Proximal PL is satisfied when:
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  - f satisfies PL and g is constant.
  - $f = h(Ax)$ for strongly-convex h and g is indicator of polyhedral set.
  - F is convex and satisfies QG.
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- Includes dual support vector machines (SVM) problem:
  - Implies linear rate of SDCA for SVMs.
- Includes ℓ_1-regularized least-squares (LASSO) problem:
  - No need for RIP, homotopy, modified restricted strong convexity,...

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  - Gives trivial proof and is weaker than more recent conditions.
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  - Weakest condition that guarantees global minima.
- We can use the inequality to analyze huge-scale methods:
  - Coordinate descent, stochastic gradient, SVRG, etc.
- We give proximal-gradient generalization:
  - Standard algorithms have linear rate for SVM and LASSO.

Thank you!