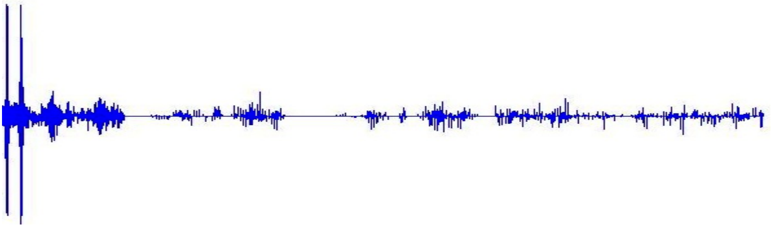
# Putting the Curvature Back into Sparse Solvers

Julie Nutini
*University of British Columbia*



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# OUTLINE

# GENERAL PROBLEM SETTING

Consider the problem

$$\underset{x\in\mathbb{R}^n}{\text{minimize}} \quad f(x) + g(x) \equiv F(x),$$

where

- $f : \mathbb{R}^n \to \mathbb{R}$ is a **convex** quadratic function, and
- $g : \mathbb{R}^n \to \mathbb{R}$ is a closed **convex** separable function with an inexpensive proximal operator.

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$\rightarrow$ $g$ is not necessarily differentiable

# EXISTING METHODS

**First-order methods**

$\rightarrow$ handle non-differentiable problems

e.g.

- Cutting Plane
- Smoothing
- Subgradient
- Bundle
- Proximal Gradient
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$\rightarrow$ can be slow to converge

# VISUALIZE

(OUR MOTIVATION . . . )

The *basis pursuit denoising problem*:

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad \frac{1}{2}\|Ax - b\|_2^2 + \lambda\|x\|_1$$

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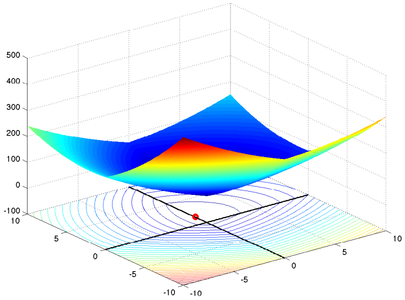


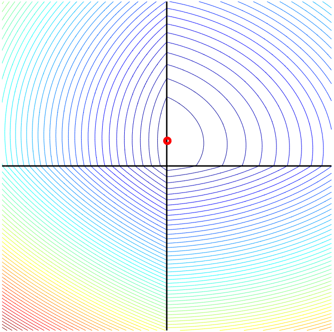

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- Exploit curvature
- Create second-order (sparse) solver
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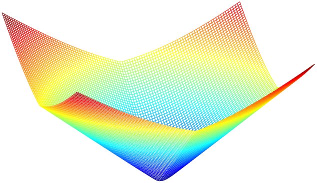

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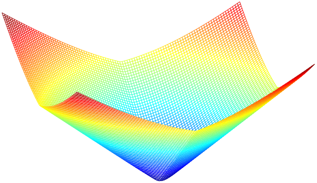

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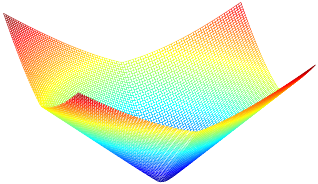

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**NOTE:** No additional storage overhead required.

Two phases, two beautiful methods …

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The *proximal operator* for a convex function $g$ is defined as

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(**Note:** When $g \equiv 0$, then just steepest descent)

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**CATCH**: CG method requires a differentiable problem.

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- **Phase 2:** Conjugate Gradient Method
  - Explore *smooth reduced subproblem* defined by the *free set*.
- Check if updated active set is *stationary*.
  - **Yes**: Continue exploring in Phase 2.
  - **No**: Define new working set in Phase 1.

# Proximal Gradient Conjugate Gradient Algorithm
# ( `pgcg` )

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Consider a quadratic bound constrained problem:

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad f(x) + \delta_\Omega(x) \equiv F(x),$$

where $\Omega = \{y : l \leq y \leq u\}$ and $\delta_\Omega(x) = \begin{cases} 0 & \text{if } x \in \Omega; \\ +\infty & \text{if } x \notin \Omega. \end{cases}$

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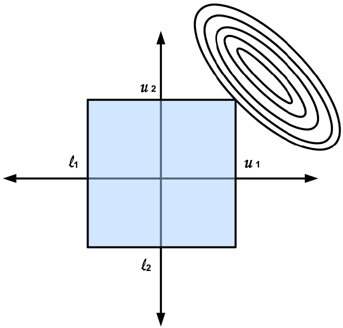

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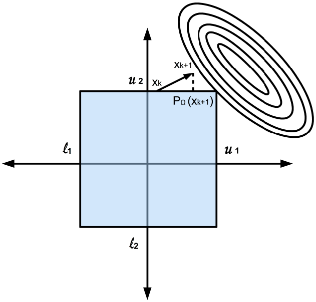
&= P_\Omega(x)
\end{aligned}$$



# THE WORKING SET

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The *active set*:

$$\mathcal{A}(x) = \{i : \partial g_i(x_i) \text{ is } \textit{not} \text{ a singleton}\}$$

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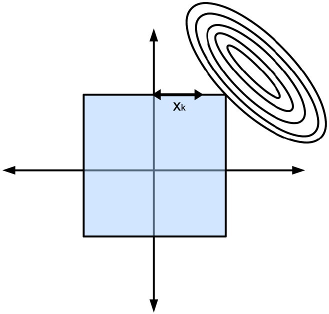

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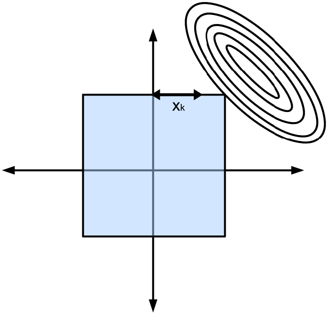


**Result:** $g$ is differentiable with respect to the free variables.

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= \quad & \ldots \text{linearize, substitute, simplify} \ldots
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$$\underset{w}{\text{minimize}} \quad \frac{1}{2} w^\top H_k w + r_k^\top w \equiv F_k(w),$$

where

- $Z_k = \mathcal{I}[:, \mathcal{F}(x_k)]$,
- $H_k = Z_k^\top H Z_k$ is a reduced Hessian, and
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Then

$$d_k = Z_k w^* = \begin{cases} w_i^*, & \text{for } i \in \mathcal{F}(x_k) \\ 0, & \text{for } i \in \mathcal{A}(x_k) \end{cases}.$$

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i.e., the set of indices corresponding to the active variables that are stationary:

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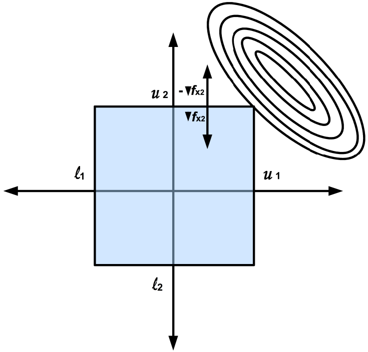

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$$y_{j+1} = \mathbf{prox}_{\alpha g(\cdot)}\big(y_j - \alpha \nabla f(y_j)\big), \quad \alpha > 0$$

   Set $x_k$ equal to first $y_j$ satisfying

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$$y_{j+1} = \mathbf{prox}_{\alpha g(\cdot)}\big(y_j - \alpha \nabla f(y_j)\big), \quad \alpha > 0$$

Set $x_k$ equal to first $y_j$ satisfying

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For fixed $\mu \in (0,1)$, find $\alpha > 0$ such that

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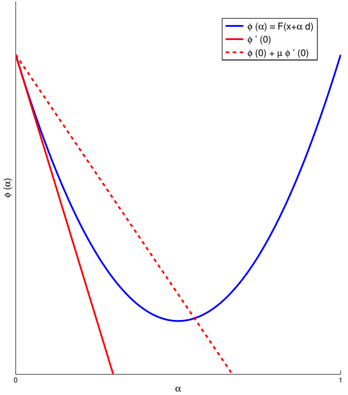

# CONVERGENCE

- A proximal gradient iteration guarantees that

$$F\left(\mathbf{prox}_{\alpha g(\cdot)}(x_k - \alpha \nabla f(x_k))\right) < F(x_k), \quad \text{for some } \alpha > 0,$$

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$\Rightarrow$ **As $F(x)$ is *non-increasing* in Phase 2, global convergence follows from Phase 1.**

$$g(\cdot) = ?$$

(Examples)

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$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad f(x) + \delta_\Omega(x) \equiv F(x), \text{ where } \Omega = \{y : l \leq y \leq u\}$$

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**Reduced subproblem:**

$$\underset{w}{\text{minimize}} \quad \frac{1}{2} w^\top H_k w + r_k^\top w \equiv F_k(w), \text{ where } r_k = Z_k^\top \nabla f(x_k)$$

This specialization is the *gradient projection conjugate gradient* method, [Moré, Toraldo, '91].

# BASIS PURSUIT DENOISING PROBLEM

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad \frac{1}{2}\|Ax - b\|_2^2 + \lambda\|x\|_1 \equiv F(x)$$

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(PROXIMAL OPERATOR)

$$\mathbf{prox}_{\lambda\|\cdot\|}(x) \quad = \quad \underset{u}{\operatorname{argmin}} \quad \lambda\|u\|_1 + \frac{1}{2}\|u - x\|_2^2$$

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$(u \neq 0)$:

$$\begin{aligned} x &= \lambda \text{sgn}(u) + u \\ \Rightarrow \quad \text{sgn}(x) &= \text{sgn}(u) \end{aligned}$$

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$(u = 0)$:

$$\begin{aligned} x \in \lambda\partial(|u|) &= [-\lambda, \lambda] \\ \therefore \quad \text{if } x \in [-\lambda, \lambda] &\Rightarrow u = 0 \end{aligned}$$

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$$\left[\mathbf{prox}_{\lambda|\cdot|}(x)\right]_i = \operatorname{sgn}(x_i) \cdot \left[|x_i| - \lambda\right]_+$$

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$$\mathcal{B}(x) = \left\{ i : i \in \mathcal{A}(x) \text{ and } [-\nabla f(x)]_i \in \partial g_i(x_i) \right\}$$

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# Basis Pursuit Denoising Problem

(Reduced subproblem)

$$\underset{w}{\text{minimize}} \quad \frac{1}{2} w^\top H_k w + r_k^\top w \equiv F_k(w),$$

$$\text{where} \quad r_k = Z_k^\top \nabla f(x_k) + \text{sgn}(Z_k^\top x_k)$$

# BASIS PURSUIT DENOISING PROBLEM

$$\underset{x\in\mathbb{R}^n}{\text{minimize}} \quad \frac{1}{2}\|Ax-b\|_2^2 + \lambda\|x\|_1 \equiv F(x)$$

**Proximal operator:**

$$\left[\mathbf{prox}_{\lambda|\cdot|}(x)\right]_i = \text{sgn}(x_i)\cdot\left[|x_i|-\lambda\right]_+$$

**Active set:**

$$\mathcal{A}(x) = \{i : x_i = 0\}$$

**Binding set:**

$$\mathcal{B}(x) = \left\{i : x_i = 0 \text{ and } [-\nabla f(x)]_i \in [-\lambda,\lambda]\right\}$$

**Reduced subproblem:**

$$\underset{w}{\text{minimize}} \quad \frac{1}{2}w^\top H_k w + r_k^\top w \equiv F_k(w),$$

where $r_k = Z_k^\top \nabla f(x_k) + \text{sgn}(Z_k^\top x_k)$.

# Numerical Examples

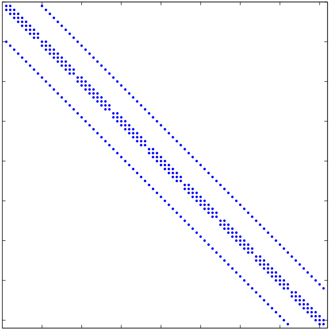
# PROBLEM SETTING

The basis pursuit denoising problem:

$$\underset{x\in\mathbb{R}^n}{\text{minimize}} \quad \frac{1}{2}\|Ax-b\|_2^2 + \lambda\|x\|_1 \equiv F(x)$$

# THEORETICAL EXAMPLE

Consider the finite-difference approximation of the Laplacian operator (discretized on a 2D grid):



$n = 10000$

condition number: 4.1336e+3

# THEORETICAL EXAMPLE

$$\underset{x\in\mathbb{R}^n}{\text{minimize}} \quad \frac{1}{2}x^\top Hx + b^\top x + 0.3\|x\|_1$$

Matrix-vector products:

| $\tau_{\text{opt}}$ | $10^{-2}$ | $10^{-4}$ | $10^{-6}$ |
|---|---|---|---|
| PG | 193 | 795 | 1498 |
| `pgcg` | 76 | 133 | 170 |

# Seismic Data Interpolation

Find sparse representation of $y$ in curvelet operator $C$:

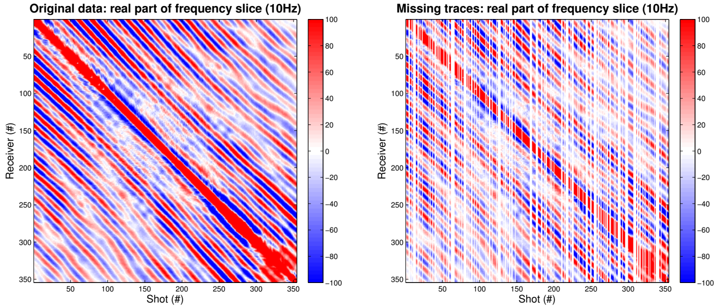
$$\underset{x\in\mathbb{R}^m}{\text{minimize}} \quad \frac{1}{2}\|RMC^\top x - y\|_2^2 + \lambda\|x\|_1,$$

where

- $RM$ is a restriction matrix operator
  - restricts to 60% original data set
- $y$ is the vectorized restricted data

# Seismic Data Interpolation

- Frequency slice (10 Hz) for sequential source acquisition from Gulf of Suez
  - e.g., [ Kumar, Aravkin, and Herrmann, 2012 ]



# SEISMIC DATA INTERPOLATION

Compare to SPGL1, which solves

$$\underset{x\in\mathbb{R}^m}{\text{minimize}} \quad \|x\|_1 \quad \text{s.t. } \|RMC^\top x - y\|_2 \leq \sigma. \tag{1}$$

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To ensure a valid comparison, we choose regularization parameter

$$\lambda = \|C(RM)^\top r_\sigma\|_\infty,$$

where $r_\sigma$ is the residual corresponding to the solution of (1) found by SPGL1.

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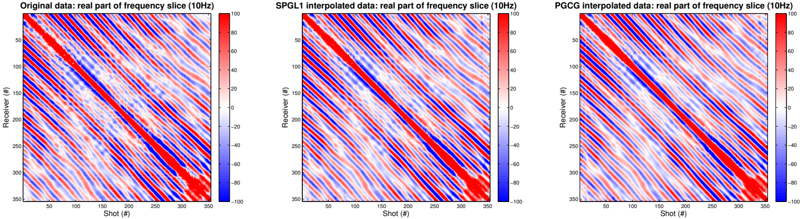

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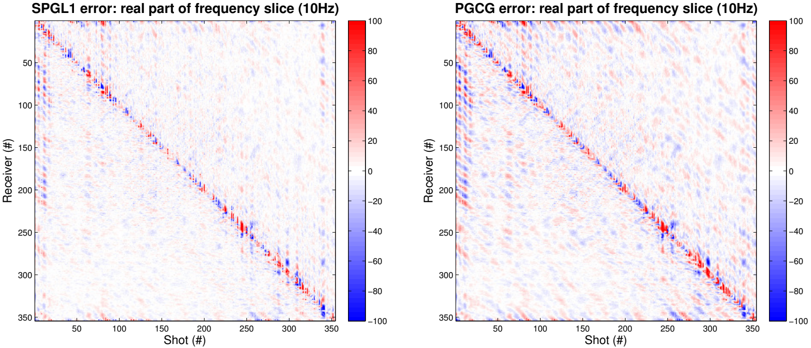


| Matrix-vector products | | |
|---|---|---|
| | $RMC^\top$ | $(RMC^\top)^\top$ |
| spgl1 | 139 | 102 |
| pgcg | 147 | 148 |

# FUTURE WORK

- A formal proof of convergence
- Using alternative methods for Phase 1
  - e.g., FISTA [Beck and Teboulle, '09]
- Alternative exit conditions
  - e.g., inside CG
- Continuation on $\lambda$, where $\lambda_k \to \lambda$
- Other interesting $g$
  - e.g., $g(z) = \|z\|_1$, where $z$ is complex valued

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THANK YOU!

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