Let's Make Block Coordinate Descent Go Fast!

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Why Block Coordinate Descent?

- Block coordinate descent methods are key tools in large-scale optimization.
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  - → Low memory requirements.
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→ **Any** improvements on convergence will affect many applications.

This Work

- We propose 4 ways to speed up Block Coordinate Descent (BCD) methods:

  1. New **greedy** block selection rules.
  2. New **second-order** update rule.
  3. New **exact** update rule for LASSO and SVMs.
  4. New **exact** update rule for graph-structured problems.

Block Coordinate Descent for Large-Scale Optimization

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→ E.g., gradient descent update $d^k = -\alpha_k \nabla_{b_k} f(x^k)$ for some $\alpha_k > 0$.

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- Assume that f is L_b-block-wise Lipschitz continuous,

$$\|\nabla_b f(x + U_b d) - \nabla_b f(x)\| \leq L_b \|d\|, \quad \text{for all } d.$$

  - → If f is twice-differentiable, this is equivalent to $\nabla^2_{bb} f(x) \preceq L_b \mathbb{I}$ for each block b.

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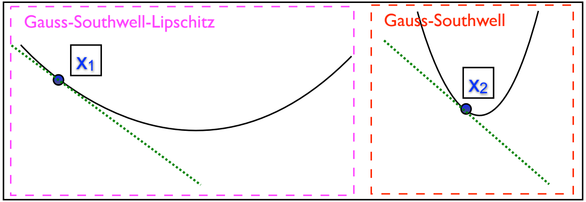

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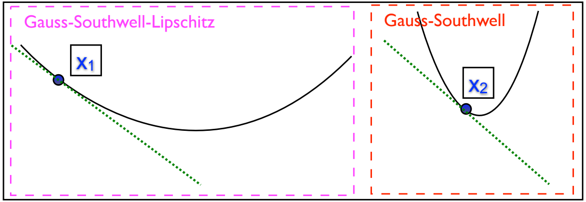

  - → Incorporates Lipschitz information in the rule.
  - → Equivalent to MI for quadratics.

Block Gauss-Southwell-Lipschitz

- As an obvious extension of the GSL rule to the block setting, we propose the **Block Gauss-Southwell-Lipschitz** (BGSL) rule:

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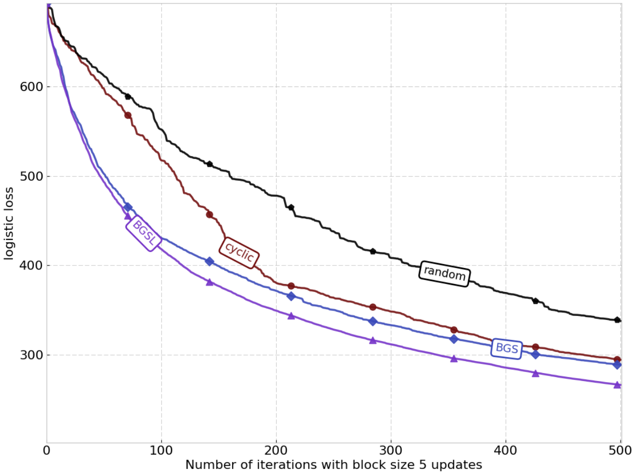
→ Unlike GSL, not equivalent to the MI rule for quadratic functions.

Experiment: L2-Regularized Logistic Regression

- Comparing block selection rules using fixed blocks.



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→ Equivalent to the MI rule for quadratics.

→ May be difficult to find Hessian bounds H_b, depends on how we define blocks.

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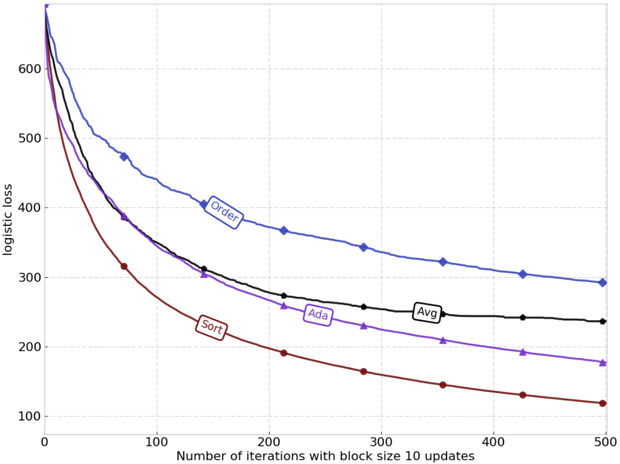
- Fixed blocks we could use Lipschitz constants to help determine the partition.

Experiment: L2-Regularized Logistic Regression

- Comparing block partitioning strategies using BGSD rule.



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$\rightarrow$ We consider a Newton-style method based on a cubic regularization framework.

Cubic Regularization Updates

- While gradient-style methods are based on a quadratic upper-bound,

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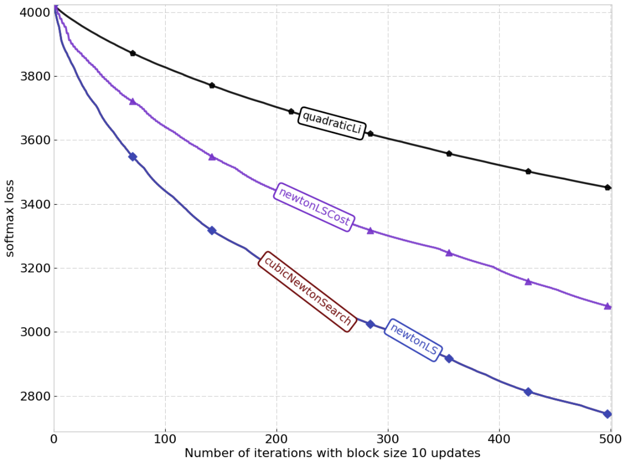
  - → Guaranteed to decrease the objective without needing extra objective function evaluations required for a line search.

Experiment: Multi-class Logistic Regression

- Comparing update rules using variable blocks with greedy block selection.



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$\rightarrow$ Possible to get superlinear convergence for problems with certain structures.

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Experiment: Dual SVM

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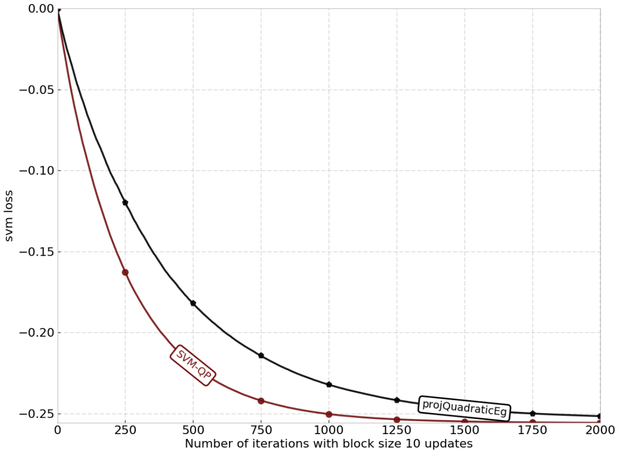

Message-Passing for Sparse Quadratics

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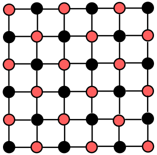

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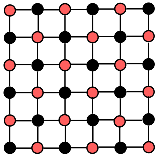

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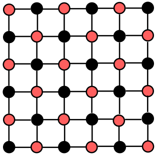
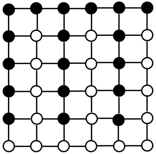
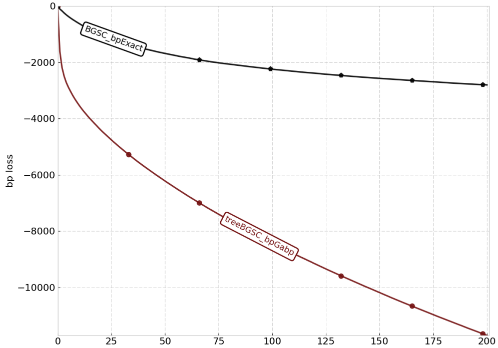


  - $\rightarrow$ Exploit connection to Gaussian Markov random fields, update tree-structured blocks in $O(M)$ using Gaussian belief propagation.
    - For lattice-structured graphs, can use blocks of size $n/2$ in $O(n)$.
    - Maintains modelling dependencies.



Experiment: Sparse Quadratic Problem

- Comparing exact updates using variable blocks with greedy selection.



- Exact solver uses $M = 8$, Gaussian belief propagation method uses $M = 8^3$.
- NP-hard to choose best “tree-structure” block.
  - $\rightarrow$ Use approximation method that performs substantially better than BGSD.

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  - Finite convergence with variable blocks and exact updates.
- Propose optimal block update strategy for sparse quadratic problems.
  - Use “tree-structured” blocks.
  - Exploits Gaussian belief propagation algorithm developed for GMRFs.
  - Requires linear time in block size.