Graphs and Combinatorics (2015) 31:393–405
DOI 10.1007/s00373-015-1543-4

ORIGINAL PAPER

Monitoring the Plane with Rotating Radars

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Received: 2 March 2014 / Revised: 15 December 2014 / Published online: 26 February 2015

Abstract Consider a set P of n points in the plane and n radars located at these points. The radars are rotating perpetually (around their centre) with identical constant speeds, continuously emitting pulses of radio waves (modelled as half-infinite *rays*). A radar can "locate" (or detect) any object in the plane (e.g., using *radio echo-location* when

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its ray is incident to the object). We propose a model for monitoring the plane based on a system of radars. For any point p in the plane, we define the *idle time* of p, as the maximum time that p is "unattended" by any of the radars. We study the following monitoring problem: what should the initial direction of the n radar rays be so as to minimize the maximum idle time of any point in the plane? We propose algorithms for specifying the initial directions of the radar rays and prove bounds on the idle time depending on the type of configuration of n points. For arbitrary sets P we give a $O(n \log n)$ time algorithm guaranteeing a $O(1/\sqrt{n})$ upper bound on the idle time, and a $O(n^6/\ln^3 n)$ time algorithm with associated $O(\log n/n)$ upper bound on the idle time. For a convex set P, we show a $O(n \log n)$ time algorithm with associated $O(1/n)$ upper bound on the idle time. Further, for any set P of points if the radar rays are assigned a direction independently at random with the uniform distribution then we can prove a tight $\Theta(\ln n/n)$ upper and lower bound on the idle time with high probability.

Keywords Convex · Detection · Idle time · Monitor · Orientation · Patrol · Plane · Points · Radar · Random · (Light) Ray

1 Introduction

Radar systems are able to detect, locate and identify (stationary and moving) objects located at great distances and in various kinds of weather conditions. They offer vast potential for effective monitoring of objects in direct sight located within a terrain.

In our setting we view the plane or parts thereof as a critical region all of whose points need to be monitored for important activities (such as animal migration, military activity, navigation guidance, weather condition reporting, etc.) taking place. It is required that specific events that may occur at any location in the plane be detected, located and reported by at least one of the sensing radars. We assume the positions of radars are fixed, for example, the radar infrastructure is already built. Our purpose is to use the "rotating radar" as a paradigm for studying combinatorial aspects of surveillance and understanding the limitations and capabilities of monitoring the plane. It is assumed that the ray of the radar can reach any point at any distance in the plane, if appropriately rotated towards this point, and that the radars have identical rotating speeds. This requires to determine in an effective manner the initial direction of the radar rays so as to optimize the time a point in the plane is left unattended by a radar. We are concerned primarily with providing algorithms for determining the initial direction of the radar rays so as to minimize the time a point in the plane may be left unattended.

1.1 Preliminaries and Notation

We consider n radars located at the points of a set $P := \{p_1, p_2, \ldots, p_n\}$ of n arbitrary points in the plane. The radars are rotating perpetually at constant identical speeds. Let the initial directions of the radar rays be $\alpha_1, \alpha_2, \ldots, \alpha_n$, with the understanding that the ith radar has initial direction measured by an angle of α_i, $0 \le \alpha_i < 2\pi$, and rotates at constant (angular) speed perpetually thereafter, for $i = 1, 2, \ldots, n$. At all times during their rotation the radars are emitting a half-infinite ray, which for our purposes can be considered to be a semi-line emanating from the point where the radar is located.

For any set $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of initial directions of the rays of the n radars we define the *idle time* of an arbitrary point p, denoted by $\mathscr{I}(P, \Pi, p)$, as the maximum time that p may be left unattended by any of the radars. Finally, we define the idle time of the system of radars as

$$\mathscr{I}(P) := \inf_{\Pi} \sup_{p} \mathscr{I}(P, \Pi, p),$$

where $\inf_\Pi$ ranges over all possible initial directions Π of radar rays and $\sup_p$ over all possible points p in the plane.

In what follows we will study our problem using an equivalent formulation based on floodlights. Floodlights have been a source of several intriguing problems in discrete and combinatorial geometry. By a *floodlight* f we mean a beam with a beam-width ϕ located at a point p of the plane called its *apex* (e.g., see [13]). Hence, the floodlight with apex p illuminates a wedge delimited between two rays (inclusively) with the common vertex p and angles α_p (the *starting ray*) and $\alpha_p + \phi$ (the *ending ray*). In this setting, it is worth mentioning the following illumination theorem from [4]:

Theorem 1 *Let P be a collection of n points in the plane, and $f_1, \ldots, f_n$ a set of floodlights with beam-widths $\phi_1, \ldots, \phi_n < \pi$, such that $\phi_1 + \phi_2 + \cdots + \phi_n \geq 2\pi$. Then there is an $O(n \log n)$ algorithm which assigns floodlights to points in P and orients them in such a way that the plane is completely illuminated.*

Note this theorem gives the directions of static (i.e., non-rotating) floodlights.

In this paper, we assume that floodlights rotate at the same speed and direction starting from their initial directions. The case when the floodlights rotate in different directions is briefly considered in Sect. 1.3. An *initial configuration* of floodlights is given by the triple $(P, \{\alpha_p\}_{p\in P}, \phi)$, where P is a finite set of points in the plane each containing the apex of a floodlight, α_p an initial direction angle of the starting ray of the beam of the floodlight with apex p, and $\phi \leq 2\pi$ is the beam-width of all floodlights. The sequence $\{\alpha_p\}_{p\in P}$ is called the *initial configuration* of the floodlights. Given the initial configuration, the *configuration at time t* is obtained from the initial one by rotating each floodlight counterclockwise by angle t. A configuration *covers* (illuminates) the plane, if each point of the plane is illuminated by at least one floodlight (not necessarily always the same) at all times during the rotation. An initial configuration is *valid*, if for every $t \geq 0$, the configuration at time t covers the plane.

Given a finite set of points P, $\Phi(P)$ is the infimum over all beam-widths ϕ such that there exists an initial configuration $(P, \{\alpha_p\}_{p\in P}, \phi)$ that is valid. A useful observation first proved in [10] (Theorem 8) is that for any set P of points in the plane the idle time definition $\mathscr{I}(P)$ and the angular floodlight definition $\Phi(P)$ are identical.

1.2 Related Work

The authors of [4] study the floodlight illumination problem but their results are not applicable since they consider only static (i.e., non-rotating) floodlights. There are several related papers on floodlight illumination. Steiger and Streinu [15] is concerned with the decision problem of illuminating a given wedge in the plane by n floodlights. Obermeyer et al. [12] develops an algorithm for a group of guards statically positioned in a non-convex polygonal environment with holes for solving the *Searchlight Scheduling Problem*: the objective of the proposed algorithm is to compute a schedule to rotate a set of searchlights in such a way that any target in an environment will necessarily be detected in finite time. Fusco and Gupta [8] is concerned with placement and orientation of rotating directional sensors so as to optimize some function of the "dark" time of the given points.

The first paper to investigate a similar problem concerning continuous coverage of a finite or infinite domain using rotating floodlights (in that paper called antennae) of a given beam width is [10]. In [10], the authors studied the Rotating Antennae Coverage Problem concerning uninterrupted coverage of a region in the plane by rotating antennae and gave algorithms for determining the initial direction of the antennae and analyze the resulting beam-width/range tradeoffs for ensuring continuous coverage of a given region or line in the plane with rotating antennae of given fixed beam-width and range. Both instances of the problem were considered: (1) rotating antennae with finite range and beam-width, and (2) rotating antennae with infinite range and given finite beam-width (which is equivalent to the floodlight formulation of our problem). In addition, to several results for uninterrupted coverage, they also proved that an angle of $3\pi/n$ is necessary and sufficient for $n \geq 2$ antennae all located on a line to cover a half-plane defined by this line at all times. This easily implies that an angle of $6\pi/n$ is sufficient for $n \geq 2$ antennae on a line to cover the plane at all times. They also proved that $\Phi(P) = \pi$, for any set P of three points in the plane. However [10] does not consider algorithms for the more general case of arbitrary sets of points. Another related paper is [2] which studies the problem of finding the minimum angle α such that one can install at each point of a pointset F a stationary rotating floodlight with illumination angle, initially oriented in a suitable direction, in such a way that, at all times, every target point of another pointset P is illuminated by at least one floodlight.

The problem of monitoring a region to minimize idleness is studied in the mobile robot literature under the name *patrolling*. It is usually defined as a perpetual process performed by mobile robots in a static or in a dynamically changing environment. Patrolling has been studied intensively in robotics (see [1,5–7,9,11,16]), where it is often viewed as a form of *coverage*. It is defined as the act of surveillance consisting of walking around an area in order to protect or supervise it. The frequency of visits as a criterion for measuring the efficiency of patrolling was first introduced in [11] where it was called *idleness*. For a survey of diverse approaches to patrolling based on idleness criteria we refer to [1].

Standard investigations on patrolling have focused on one-dimensional models by ensuring that a boundary encircling a given two-dimensional domain is patrolled by robots perpetually moving along the boundary. Despite the fact that such one-dimensional models provide adequate solutions for patrolling the boundary and thus monitoring potential incursions, they do nothing to address monitoring of the interior area delimited by this boundary. Thus the model studied in this paper is more suitable for monitoring in the two dimensional plane.

Table 1 The time complexity of finding the valid initial direction with given beam-widths for n points in the plane

Point-set	Initial direction	Beam-width	Complexity
Arbitrary	Deterministic	$O(\log n/n)$	$O(n^6/\ln^3 n)$
Arbitrary	Deterministic	$O(1/\sqrt{n})$	$O(n \log n)$
Arbitrary	Random[a]	$\Theta(\log n/n)$	$O(n)$
Convex	Deterministic	$O(1/n)$	$O(n \log n)$

The first column describes the configuration of points, the second the type of initial direction, the third the beam-width and the last the time complexity for finding such an initial direction

[a] The initial direction of floodlights is chosen independently and uniformly at random and is valid with high probability

1.3 Outline and Results of the Paper

We give algorithms providing the initial directions of the radar rays and provide various bounds for several types of configurations of a set P of n points in the plane. Since the rotation of floodlights is a continuous process, Sect. 2 discusses a discretization of the rotation of floodlights.

In Sect. 3, we prove an $O(\log n/n)$ upper bound on $\Phi(P)$ for any set P of n points in the plane. However, since this algorithm for finding the initial direction of floodlights with beam-width $\Theta(\log n/n)$ is not very efficient, we will also present a fast $O(n \log n)$ algorithm for floodlights with beam-width $\Theta(1/\sqrt{n})$ and a randomized $O(n)$ algorithm for floodlights with beam-width $\Theta(\log n/n)$. In Sect. 4, we consider points in convex position; we give an $O(1/n)$ upper bound on the idle time for any set P of n points in convex position. We conclude in Sect. 5 with several open questions. Results of the paper are summarized in Table 1.

2 Discretizing the Rotation of Floodlights

Rotation of floodlights is a continuous process. In order to study the problem, we introduce a discrete model of the floodlights' rotation. Before proving the upper bound we define several concepts concerning the relation between discrete and continuous movement of the floodlights.

Definition 1 (*Discretely rotating floodlight*) Let p be a point in the plane, and let k and m be positive integers. Let $b_0, \ldots, b_{k-1}$ be beams originating at p with beam-width $2\pi/k$ such that the starting ray of b_i has angle $2\pi i/k$. We will assume that for any integer p, $b_p = b_{p \bmod k}$. Note that the beams together illuminate the whole plane. These beams do not rotate, but they do turn on and off. A *discretely rotating floodlight at* p *with beam-width* $2\pi m/k$, *step-width* $2\pi/k$ *and initial direction* $i \in \{0, \ldots, k-1\}$ is the configuration of the beams such that for any integer T, at any time $t \in [T/k, (T+1)/k)$, only beams $b_{T+i}, b_{T+i+1}, \ldots, b_{T+i+m-1}$ are on. Hence, this discretely rotating floodlight illuminates the region of the plane between rays with

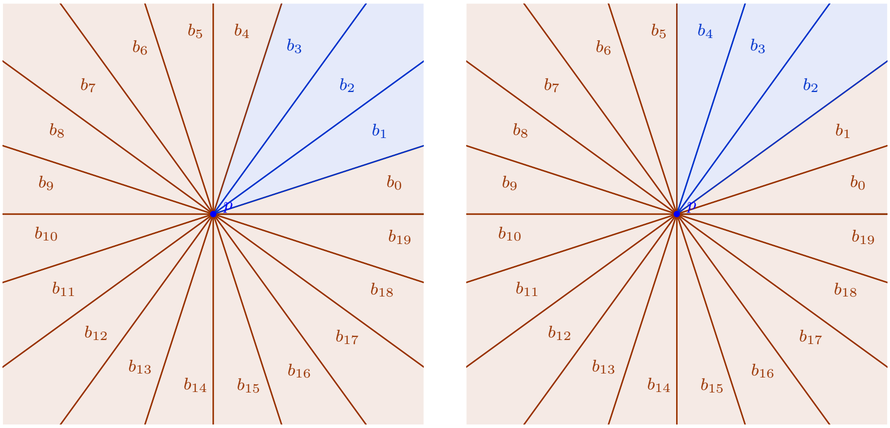


Fig. 1 Discretely rotating floodlight at a point p with beam-width $3\pi/10$, step-width $\pi/10$ and initial direction $i = 1$. On the *left* the floodlight at times $0 \leq t < 1/20$ and on the *right* the floodlight at times $1/20 \leq t < 1/10$

angles $2\pi(i + \lfloor tk \rfloor)/k$ and $2\pi(i + m + \lfloor tk \rfloor)/k$ at time t (see Fig. 1 for an illustration of the concepts defined).

In terms of covering, continuously rotating floodlights and discretely rotating floodlights are equivalent up to constant factors of their beam-widths as stated in the following theorem.

Theorem 2 *Let p be a point in the plane, and let k and m be positive integers. A floodlight with beam-width $\phi \leq 2\pi/k$ rotating (continuously) around a point p can be covered by a discretely rotating floodlight at p of beam-width $6\pi/k$ and step-width $2\pi/k$. Conversely, a discretely rotating floodlight at a point p with beam-width $2\pi m/k$ and step-width $2\pi/k$ can be covered by a continuously rotating floodlight at p with beam-width $2\pi(m + 1)/k$.*

Proof Consider a continuously rotating floodlight f with beam-width $\phi \leq 2\pi/k$ at a point p and initial direction α. Let f' be the discretely rotating floodlight at p with beam-width $6\pi/k$, step-width $2\pi/k$ and initial direction i such that beams b_i and b_{i+1} of f' cover f at time $t = 0$. Since the beam-width of the union of these two beams is larger than the beam-width of f, such an i must exist. At time $0 \leq t \leq 2\pi/k$, f has not moved by more than $2\pi/k$ from its initial direction, so it is covered by b_i, b_{i+1} and b_{i+2}. At time $t = 2\pi/k$, f' switches off b_i and on b_{i+3} and f has moved at least $2\pi/k$, but not more than $4\pi/k$, hence f is covered by beams b_{i+1}, b_{i+2} and b_{i+3} at time $2\pi/k \leq t \leq 4\pi/k$. This argument can be extended to all other time intervals, in particular, at any time t, f is covered by beams $b_{i+\lfloor kt \rfloor}$, $b_{i+\lfloor kt \rfloor+1}$, $b_{i+\lfloor kt \rfloor+2}$, which are turned on.

Consider a discretely rotating floodlight f with beam-width $2\pi m/k$, step-width $2\pi/k$ and initial direction i. Let f' be the continuously rotating floodlight with beam-width $2\pi(m + 1)/k$ and initial direction $2\pi(i - 1)/k$. Note that f' covers beams $b_{i-1}, \ldots, b_{i+m-1}$ of f at time $t = 0$, and beams $b_i, \ldots, b_{i+m-1}$ at time $0 \leq t < 2\pi/k$, which are all beams turned on. For any integer T, at time $2\pi T/k \le t < 2\pi(T+1)/k$, f' covers beams $b_{i+T}, \ldots, b_{i+m-1+T}$, which are all beams turned on at this time. □

In the rest of the paper we will prove all upper bounds for discretely rotating floodlights. By Theorem 2, these bounds extend to continuously rotating floodlights with double beam-width.

3 Algorithms on Arbitrary Sets of Points

We start with the $O(n \log n)$ algorithm for finding the initial direction of floodlights with beam-width $O(1/\sqrt{n})$.

Theorem 3 *There is an $O(n \log n)$ algorithm that for any set P of n points in the plane, finds a valid initial direction of discretely rotating floodlights with beam-width and step-width $O(1/\sqrt{n})$ at points in P.*

The proof uses the following corollary of the main illumination theorem from [4]:

Corollary 1 *Let P be a collection of $n \ge 3$ points in the plane, each with floodlight having beam-width $2\pi/n$. Then there is an $O(n \log n)$ algorithm which orients the floodlights in such a way that the plane is completely illuminated.*

Proof Let $m = \lfloor \sqrt{n} \rfloor$. Pick m mutually disjoint subsets $S_0, \ldots, S_{m-1}$ of P of size m. For each $j = 0, \ldots, m-1$, find the direction of static floodlights with the beam-width $2\pi/m$ at points in S_j that illuminate the plane using the $O(m \log m)$ algorithm from Corollary 1. For each point p in S_j, define the discretely rotating floodlights with beam-width $4\pi/m$, step-width $2\pi/m$ and initial direction i such that at time $t = 2\pi j/m$, the floodlight covers the static floodlight at p. It follows that for every $T = 0, \ldots, m-1$, the floodlights in S_T cover the plane at any time $2\pi T/m \le t < 2\pi(T+1)/m$, hence, the place is covered at all times. The time complexity of this algorithm is $O(mm \log m) = O(n \log n)$. □

Next, we will present an $O(n^6/\ln^3 n)$ algorithm for finding the initial direction of floodlights with beam-width $\Theta(\log n/n)$. We will need the following result about partitioning a plane into regions by a collection of lines.

Lemma 1 (Chapter 8.3 in [3]) *Given n lines in the plane, there is an $O(n^2)$ algorithm, which reports all regions (faces) determined by these lines.*

Theorem 4 *There is an $O(n^6/\ln^3 n)$ algorithm that for any set P of $n \ge 2$ points in the plane, finds a valid initial direction of discretely rotating floodlights at points in P with beam-width ϕ and step-width ϕ, where $\phi \in O(\log n/n)$.*

Proof Let $P = \{p_1, p_2, \ldots, p_n\}$ and let $k \le \lfloor n/(18 \ln n) \rfloor$. At each $p \in P$, construct a discretely rotating floodlight with beam-width π/k and step-width π/k. For each point $p \in P$ the boundaries of its $2k$ beams is the union of k lines through p. The union of these boundary lines is a collection of at most kn lines which partition the plane into at most

$$\binom{kn+1}{2} + 1 \le k^2 n^2$$

regions. We denote this set of regions by $\mathscr{R}$. Note that each region is included in exactly one of the $2k$ beams of each floodlight. Also note that the plane is covered at time t by the floodlights if and only if each region is included in at least one beam that is on at time t. Since floodlights will return to their initial direction at time $t = 1$, the plane is covered at all times, if it is covered at times $t \in \mathscr{T} = \{0, \pi/k, 2\pi/k, \ldots, (2k-1)\pi/k\}$. To verify that the plane is covered, we define the coverage function C on a set of floodlights S which returns all pairs $(r, t) \in \mathscr{R} \times \mathscr{T}$ such that region r is covered by at least one of the floodlights in S at time t. In what follows we assign the initial directions to floodlights such that the coverage function of all n floodlights returns exactly $2k|\mathscr{R}|$ pairs, i.e., all regions are covered at all times in $\mathscr{T}$. The main algorithm is as follows.

1. Assign the initial direction of the floodlight at p_1 arbitrarily.
2. Suppose we have already assigned the initial direction of floodlights at points $p_1, \ldots, p_{j-1}$, $j < n$. Pick the initial direction of the floodlight at p_j such that the number of pairs returned by the coverage function of the first j floodlights is maximized.

Let C_j denote the set of pairs returned by the coverage function for the first $j \leq n$ floodlight configured by the algorithm. It is easy to see that $|C_1| = |\mathscr{R}|$ ("a floodlight illuminates all regions in one rotation"). In addition, we will show that $|C_j| \geq |C_{j-1}| + |\mathscr{R}| - \frac{|C_{j-1}|}{2k}$. Note that $2k$ possible initial directions of the floodlight at p_j partitions $\mathscr{R} \times \mathscr{T}$ into $2k$ subsets $S_0, \ldots, S_{2k-1}$ of the same size $|\mathscr{R}|$, where S_i is the subset of pairs covered by the floodlight at p_j with initial direction i. The algorithm will choose the initial direction i of the floodlight at p_j so that the size of the intersection of C_{j-1} and S_i is minimized, i.e., it has size at most $\frac{|C_{j-1}|}{2k}$. Hence, the number of newly covered pairs by the floodlight at p_j is at least $|\mathscr{R}| - \frac{|C_{j-1}|}{2k}$, and the bound follows. Now using that $|C_1| = |\mathscr{R}|$ (for the base case) and this inequality (for the inductive step), it is easy to prove by induction on j that $|C_j| \geq 2k|\mathscr{R}|(1-(1-\frac{1}{2k})^j)$.

If we show that $(1-\frac{1}{2k})^n < \frac{1}{2k|\mathscr{R}|}$, then $|C_n| > 2k|\mathscr{R}| - 1$, and since $|C_n|$ is an integer, we have that $|C_n| = 2k|\mathscr{R}|$. To show this observe that for $n \geq 2$,

$$\left(1 - \frac{1}{2k}\right)^n \leq e^{-\frac{n}{2k}} \leq e^{-9\ln n} < \frac{18^3 \ln^3 n}{2n^5} \leq \frac{1}{2k^3n^2} \leq \frac{1}{2k|\mathscr{R}|}, \tag{1}$$

where the second and fourth inequalities follow from the bound on k.

Implementation and complexity. By Lemma 1, the set of all regions $\mathscr{R}$ determined by kn lines can be constructed in time $O(k^2n^2)$. To represent the result returned by the coverage function, we will use a boolean array with an entry for each pair in $\mathscr{R} \times \mathscr{T}$. In the jth iteration of the algorithm, we count for each of $2k$ initial directions of the floodlight at point p_j how many new pairs will be returned by the coverage function as follows. For each region in $\mathscr{R}$, we determine in $O(1)$ time in which beam of the floodlight at p_j it lies, and hence, at what time it will be covered by this floodlight. After that we pick the best direction and update the array correspondingly. This takes time $O(|\mathscr{R}|k) = O(k^3n^2)$, for $j \geq 2$ and time $O(|\mathscr{R}|) = O(k^2n^2)$ for $j = 1$, since we can pick any initial direction. Since we have n points, the total running time is $O(k^3n^3) = O(n^6/\ln^3 n)$. □

We remark that the constant 9 in the previous theorem can be replaced by any $c > 10$, and the theorem still holds for n sufficiently large (depending on c).

Next, we will modify the argument in the previous proof to obtain a randomized $O(n)$ algorithm for floodlights with beam-width $\Omega(\log n/n)$. This algorithm picks the initial directions of all floodlights at random. We have the following theorem.

Theorem 5 *Let P be a set of $n \geq 2$ points in the plane. Let $k \leq \lfloor n/(20 \ln n) \rfloor$. If the initial directions of discretely rotating floodlights at P with beam-width and step-width π/k is chosen randomly and independently, then the plane is covered at all times with high probability. If all n floodlights are placed at the same point and $k \geq \lceil 2n/\ln n \rceil$ and the initial directions of discretely[1] rotating floodlights with beam-width and step-width π/k are chosen independently and uniformly at random, then the plane is not covered at all times with high probability.*

Proof As in the proof of Theorem 4, sets C_j of pairs $(r, t) \in \mathscr{R} \times \mathscr{T}$ returned by the coverage function, $j = 1, \ldots, n$. We still have $|C_1| = |\mathscr{R}|$, however, for $j \geq 2$, we can show by the same argument as in the previous proof that the expected size of C_j is $E[|C_j|] = E[|C_{j-1}|] + |\mathscr{R}| - \frac{E[|C_{j-1}|]}{2k}$, and hence also $E[|C_n|] = 2k|\mathscr{R}|(1 - (1 - \frac{1}{2k})^n)$. We have

$$\Pr(|C_n| = 2k|\mathscr{R}|) \geq E[|C_n|] - 2k|\mathscr{R}| + 1 = 1 - 2k|\mathscr{R}| \left(1 - \tfrac{1}{2k}\right)^n .$$

Similarly as in (1), we have

$$\left(1 - \frac{1}{2k}\right)^n < \frac{1}{2kn|\mathscr{R}|}$$

for $n \geq 2$. Notice that the extra n in the denominator on the right hand side of the inequality comes from replacing constant 18 with 20 in the assumption on k. Therefore, $\Pr(|C_n| = 2k|\mathscr{R}|) > 1 - 1/n$, i.e., the floodlights cover the plane at all times with probability going to 1 as n goes to infinity.

To prove the second part of the statement, suppose $k \geq \lceil 2n/\ln n \rceil$. Since all floodlights are placed at the same point, it is enough to show that the initial configuration, when chosen independently and uniformly at random, does not cover the plane with high probability. Indeed, if the initial configuration covers the whole plain, then the plain remains covered at all times. The beams of floodlights partition the plane into $2k$ regions. Let X be the random variable counting the number of uncovered regions by floodlights in the initial configuration. We have that

$$E[X] = 2k \left(1 - \frac{1}{2k}\right)^n \geq \frac{2n^{3/4}}{\ln n}$$

for sufficiently large n, since $E[X]$ approaches $\frac{4n^{3/4}}{\ln n}$ as n goes to infinity. Since changing the initial direction of one floodlight changes the number of uncovered regions in the initial configuration by at most one, by Azuma's inequality, we obtain

[1] A similar result can be proved for continuously rotating floodlights using a result of Penrose [14].

$$\Pr(|X - E[X]| < t) \geq 1 - 2e^{-t^2/(2n)}.$$

Now if we set t to $E[X]$, we have

$$Pr(X \geq 1) \geq 1 - 2e^{-\frac{E[X]^2}{2n}} \geq 1 - 2e^{-\frac{2n^{1/2}}{\ln^2 n}}$$

which approaches 1 as n goes to infinity. □

We again remark that the constant 20 in the first part of the previous theorem can be replaced by any $c > 12$ and $1/2$ in the second part can be replaced by any $c < 1$, and the theorem still holds for n sufficiently large (depending on c).

4 Algorithms for Points in Convex Position

In this section we restrict ourselves to the set P of n points in convex position.

Without loss of generality, we will assume that at least $\lfloor n/2 \rfloor$ of the points of P lie above the x-axis and at least $\lfloor n/2 \rfloor$ below the x-axis. We need additional concepts of sweep line, and east and west subsets of a convex set.

Definition 2 (Sweep line hitting a set) Given an angle α, let $\text{dir}(\alpha)$ be the unit vector with angle α. Given a set S of points in the plane, and an angle α, the *sweep line with angle* α *hits* S *at* p if $p \in S$ and if ℓ is a translation of the sweep line such that $p \in \ell$ and H is the half-plane with boundary ℓ containing point $p + \text{dir}(\alpha - \pi/2)$, then $S \subseteq H$. We also say that p is a *hitting point* of the sweep line and S. Note that p does not have to be unique.

Definition 3 (East and west subsets) Given a convex set of points S, let the *north pole* of P, denoted s_0, be the point at which the sweep line with angle 0 hits P (and has the smallest x-coordinate if there are several such points). Let $s_1, s_2, \ldots, s_{n-1}$ be the remaining points of P as they appear along the convex hull of P starting from s_0 in the clockwise direction. The *south point* of S is defined similarly with respect to the sweep line with angle π. Suppose s_t is the south pole. The *east subset* of P is the set $\{s_0, \ldots, s_t\}$ and the *west subset* of P is the set $\{s_{t+1}, \ldots, s_{n-1}\}$.

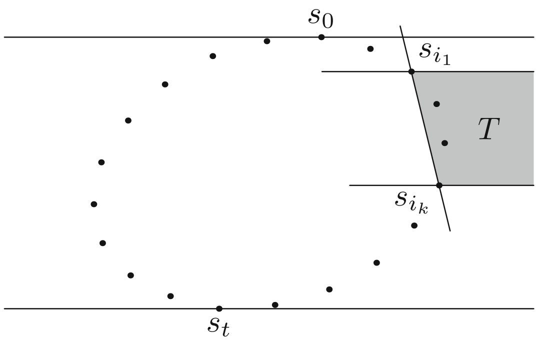
The east subset behaves very nicely with respect to sweep lines with angles between 0 and π as stated in the following lemma whose proof can be found in the Appendix.

Lemma 2 *If* $S = \{s_{i_1}, \ldots, s_{i_k}\}$*, where* $0 \leq i_1 < \cdots < i_k \leq t$*, is a subset of the east subset of* P*, then the sweep line with angle between 0 and* π *hits* S *at either* s_{i_1} *or* s_{i_k}*.*

Proof Let T be the intersection of three halfplanes:

- the halfplane with the boundary line through s_{i_1} and s_{i_k} containing $s_{i_1} + \text{dir}(0)$,
- the halfplane with the boundary line with angle 0, passing through s_{i_1} and containing s_{i_k}, and
- the halfplane with the boundary line with angle 0, passing through s_{i_k} and containing s_{i_1},

Fig. 2 The intersection of three halfplanes that contain the set S. Any sweep line with angle between 0 and π hits T at either s_{i_1} or s_{i_k}



cf. Fig. 2. It is easy to see that one of hitting points of the sweep line with angle between 0 and π and T is at either s_{i_1} or s_{i_k}. Since S is a subset T and contains s_{i_1} and s_{i_k}, the same holds true for S. □

The following key lemma shows how to illuminate a half plane using a subset of the east subset of points.

Lemma 3 *Let S be a set of points of size at least $6k - 2$ all lying below the x-axis and in the east subset of P. Then there exist initial directions of discretely rotating floodlights with the beam-width and the step-width $\phi = \pi/k$ at these points such that the half-plane above the x-axis is covered at all times.*

Proof Let $s_{m_1}, \ldots, s_{m_{6k-2}}$ be $6k - 2$ points of S such that $m_1 < \cdots < m_{6k-2}$. For every $i = 1, \ldots, 3k - 1$, set the initial direction of the floodlights at s_{m_i} and $s_{m_{6k-1-i}}$ to $i \bmod 2k$. It is enough to show that for any $t \in \{0, 1/2k, \ldots, (2k-1)/2k\}$, the half-plane above the x-axis is covered at time t. For every such t, we can choose $2k$ points $s_{l_1}, \ldots, s_{l_k}, s_{r_k}, \ldots, s_{r_1}$ out of these $6k - 2$ points such that $l_1 < \cdots < l_k < r_k < \cdots < r_1$ and for every $i = 1, \ldots, k$, the direction of the floodlights at s_{l_i} and s_{r_i} at time t is $i - 1$. Among these $2k$ points we will choose k using the following procedure. For each $i = 1, \ldots, k$, let point p_i be a hitting point of the sweep line with angle $i\phi$ and the set $\{s_{l_i}, \ldots, s_{l_k}, s_{r_k}, \ldots, s_{r_i}\}$. By Lemma 2, p_i can be chosen to be either s_{l_i} or s_{r_i}. Next, we show by induction on i that

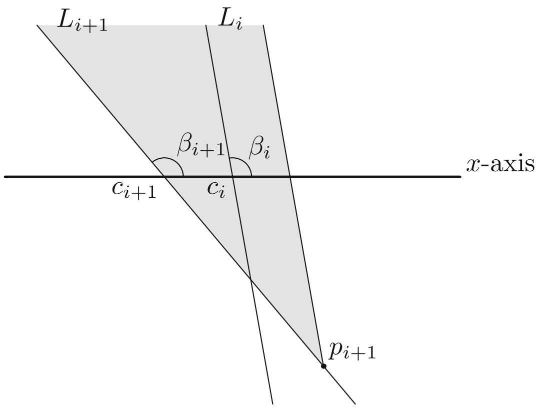
– the floodlights at points $p_1, \ldots, p_i$ cover an angle β_i with the starting ray with angle 0 (lying on the x-axis) and the ending ray with angle $i\phi$ (lying on the line L_i passing through p_i). We call the common vertex of these two rays c_i.

For $i = 1$, let L_1 be the line passing through p_1 with angle ϕ and c_1 be the intersection point of L_1 and the x-axis. Obviously, the angle β_1 with the starting ray with angle 0, the ending ray lying on L_1 and common vertex c_1 satisfies the invariant.

Suppose that the induction invariant holds for $i < k$. Let L_{i+1} be the line passing through p_{i+1} with angle $(i + 1)\phi$ and c_{i+1} be the intersection point of L_{i+1} and the x-axis. Since point p_{i+1} lies in the right closed half-plane determined by L_i, angle β_i and the floodlight at point p_{i+1} cover angle β_{i+1}, cf. Fig. 3. Hence, the first $i + 1$ floodlights cover angle β_{i+1}.

Note that if $i = k$, the above invariant implies that floodlights at $p_1, \ldots, p_k \in S$ cover the half-plane above the x-axis at time t. □

Fig. 3 Illustration how angle β_i and the flood light at point p_{i+1} cover angle β_{i+1}



We now state and prove the main theorem of this section, which is valid for a set P of n points in convex position. Recall that in this section we assume that at least $\lfloor n/2 \rfloor$ of the points of P lie above the x-axis and at least $\lfloor n/2 \rfloor$ below the x-axis.

Theorem 6 *Let P be a set of n points in the plane in convex position. Then there exists a valid initial direction of discretely rotating floodlights at points in P with beam-width and step-width $\phi = \pi / \lfloor n/24 \rfloor$, which can be found in time $O(n \log n)$.*

Proof Let P_E^- (P_W^-) be the subset of P containing all points of the east (west) subset of P that lie below the x-axis. At least one of the sets P_W^- and P_E^- contains at least $\lfloor n/4 \rfloor > 6\lfloor n/24 \rfloor - 2$ points. If it is the set P_E^-, it follows by Lemma 3, that there exist initial directions of discretely rotating floodlights with the beam-width and step-width $\phi = \frac{\pi}{\lfloor n/24 \rfloor}$ such that the halfplane above the x-axis is covered at all times. If it is P_W^-, we apply Lemma 3 on the reflection of P_W^- about the vertical line through the south pole s_t, since this reflection has the properties of the east subset. To each point $p \in P_W^-$, assign the reflection of the initial direction of the floodlight at the reflection of p, i.e., if the floodlight at the reflection of p has initial direction i, assign the initial direction $(k - i) \bmod 2k$ to the floodlight at p. Since the floodlights at the reflected points cover the halfplane above the x-axis at all times, so do the floodlights at P_W^-. By symmetry, the floodlights at points above the x-axis can be set up to cover the half-plane below the x-axis.

To find the order of points of P along the convex hull takes time $O(n \log n)$, after which assigning of initial directions can be done in linear time. □

By Theorems 2 and 6 holds for continuously rotating floodlight with beam-width 2ϕ.

5 Conclusion and Open Problems

In this paper we have investigated the idle time of rotating floodlights in the plane. We derived monitoring algorithms for radars located on arbitrary sets of points, and for points in convex position. As an interesting problem we note that no general lower bound result is known for the case of a regular n-gon other than the straightforward

$2\pi/n$. For any set of n points, we also derived the idle time $O(\ln n/n)$ when the radars are orientated randomly and independently with the uniform distribution. Further, in view of Theorem 5, $\Theta(\ln n/n)$ is also a lower bound assuming all the floodlights are located at the same point in the plane. However, in general, it is not known whether the problem of minimizing the idle time for floodlights on a general point set is NP-hard.

Aside from improving our bounds, several intriguing open problems remain. In all point-sets considered in this paper the floodlights had identical angles. An interesting problem concerns the possibility of generalizing the static floodlight coverage theorem to rotating floodlights $f_1, \ldots, f_n$ with arbitrary angle sizes $\phi_1, \ldots, \phi_n$, respectively, such that $\phi_1 + \cdots + \phi_n = c\pi$, for some constant c independent of n. Another interesting question concerns the possibility of having floodlights with arbitrary rotating speeds. Similarly, computing the idle time for the case of faulty radars, which may malfunction during the rotation, provides a challenging set of questions.

Acknowledgments Authors acknowledge partial support from NSERC, VEGA, and Conacyt. Many thanks to the anonymous referees for comments that improved the overall presentation of the paper.

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