Recognizing more unstatisfiable etc.

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1 Spectral Considerations

In this section we prove a general relationship between the size of an independent set in a graph and the eigenvalues of its adjacency matrix. Then we prove that the graphs G_F and H_F satisfy certain eigenvalue bounds with high probability. These two results show that with high probability the conclusion of Theorem 1 does not hold.

Let $G = (V, E)$ be an undirected graph, and $A = A_G$ the adjacency matrix of G. Let A_G's eigenvalues be ordered $\lambda_1 \geq \cdots \geq \lambda_n$, with $n = |V|$. We say that G is *ν-separated* if $|\lambda_i| \leq \nu\lambda_1$ for $i > 1$. We say that G is *ϵ-balanced* for some $\epsilon > 0$ if there is a real d such that the degree of each vertex is between $d(1-\epsilon)$ and $d(1+\epsilon)$.

Theorem 1.1 *If G is ν-separated and ϵ-balanced, then G contains no independent set of size $\geq (n/5) + nf(\nu, \epsilon)$ where $f(\nu, \epsilon)$ tends to 0 as ν, ϵ tend to 0.*

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We remark that this theorem can probably be greatly improved upon (see below). But this weak theorem does preclude independent sets of size $n/4$ for small ν, ϵ, and that is all we need here.

Proof Let u be a non-negative first eigenvector of A_G of unit length. Then

$$A_G = \lambda_1 u u^{\mathrm{T}} + \mathcal{E},$$

where $\mathcal{E}$ is a matrix of operator norm $\max_{i>1} |\lambda_i|$. Let S be an independent subset of vertices of G, and let $T = V \setminus S$. Let χ_S, χ_T be the characteristic functions of S, T respectively (i.e. taking the value 1 on the set and 0 outside of the set). Then

$$d(1-\epsilon)|S| \le \Big| \text{edges leaving } S \Big| = (A_G \chi_S, \chi_T) = \lambda_1 (u^{\mathrm{T}} \chi_S)(u^{\mathrm{T}} \chi_T) + \chi_S^{\mathrm{T}} \mathcal{E} \chi_T$$

$$\le \lambda_1 (u^{\mathrm{T}} \chi_S)(u^{\mathrm{T}} \chi_T) + \Big(\max_{i>1} |\lambda_i| \Big) \sqrt{|S|\,|T|}$$

Let α, β be the average values of u on S, T respectively. Since u is a unit vector, Cauchy-Schwarz implies that

$$\alpha^2 |S| + \beta^2 |T| \le \|u\|^2 = 1,$$

and hence, again by Cauchy-Schwarz

$$2\alpha |S|^{1/2} \beta |T|^{1/2} \le \alpha^2 |S| + \beta^2 |T| \le 1$$

(the weakness of this theorem undoubtedly comes from this pessimistic first inequality, which is only close to the truth when $\alpha^2|S|$ is close to $\beta^2|T|$). It follows that

$$(u^{\mathrm{T}} \chi_S)(u^{\mathrm{T}} \chi_T) = \alpha |S| \beta |T| \le \sqrt{|S|\,|T|} \Big/ 2.$$

Since λ_1 is bounded above and below by the max and min degrees, we conclude

$$d(1+\epsilon)|S| \le d(1-\epsilon)\sqrt{|S|\,|T|} \Big/ 2 + \nu d(1+\epsilon)\sqrt{|S|\,|T|}.$$

Setting $\theta = |S|/n$ we conclude that

$$\theta \le (1/2)\Big(1 + O(\epsilon) + O(\nu)\Big)\sqrt{\theta(1-\theta)}$$

for ϵ, ν small, and squaring both sides and dividing by θ we get

$$\theta \le (1/4)\Big(1 + O(\epsilon) + O(\nu)\Big)(1-\theta),$$

and the theorem follows.

□

We now turn to eigenvalue estimates for the random graphs described in section 1. We shall estimate the lower eigenvalues of G_F by the trace method. For two vertices (a_1, b_1) and (a_2, b_2) of G_F, and for a variable z, let

$$(a_1, b_1) \xrightarrow{(z,1)} (a_2, b_2)$$

be 1 or 0 according to whether or not both $a_1 \vee a_2 \vee z$ and $b_1 \vee b_2 \vee \neg z$ are clauses in F. Similarly, let

$$(a_1, b_1) \xrightarrow{(z,-1)} (a_2, b_2) = (a_2, b_2) \xrightarrow{(z,1)} (a_1, b_1).$$

So the number of edges from (a_1, b_1) to (a_2, b_2) is the number of pairs (z, e) with z a variable and $d \in \{1, -1\}$ for which

$$(a_1, b_1) \xrightarrow{(z,e)} (a_2, b_2) = 1.$$

We shall also say that such an edge has "colour" z.

We now describe the trace method. We fix a positive even integer, k. For vectors of variables, $\vec{a} = (a_0, \ldots, a_k)$, $\vec{b} = (b_0, \ldots, b_k)$, $\vec{z} = (z_1, \ldots, z_k)$, and a vector $\vec{e} = (e_1, \ldots, e_k)$ with each $e_i \in \{1, -1\}$, we set

$$\text{walk}(\vec{a}, \vec{b}, \vec{z}, \vec{e}) = \prod_{i=0}^{k-1} (a_{i-1}, b_{i-1}) \xrightarrow{(z_i,e_i)} (a_i, b_i).$$

Clearly

$$\sum \lambda_i^k = \text{Trace}(A^k) = \sum_{\substack{\vec{a},\vec{b},\vec{z},\vec{e} \\ a_0=a_k, b_0=b_k}} \text{walk}(\vec{a}, \vec{b}, \vec{z}, \vec{e}).$$

The "pure" trace method would estimate the expected value of the above trace. However, we shall need to exclude certain rare and pathological graphs before estimating the trace.

For a graph, G_F, let $d_{\max}$ and $d_{\min}$ be the maximum and minimum degrees of G_F. Let $c_{\max}$ be the maximum number of edges of any one colour incident upon an vertex.

Lemma 1.2 *For any constant $r > 0$ there is a constant M such that for a random G_F we have $c_{\max} \leq M$ with probability $1 - O(n^{-r})$.*

Proof Given a z, the edges incident with a given (a, b) of colour z is the the sum of two products, the first product being the number of variables x with $a \vee x \vee z$ a clause times the number of variables y with $b \vee y \vee \neg z$, and the second product similar. The number of such x's as above, with a and z fixed, is the number of successes of n Bernouli trials with probability $n^{-1-\gamma}$. For $t \geq 1$, t successes will occur with probability at most

$$(n-t+1)\binom{n}{t}\left(n^{-1-\gamma}\right)^t\left(1-n^{-1-\gamma}\right)^{n-t} \leq n\binom{n}{t}n^{t(-1-\gamma)} \leq n(ne/t)^t n^{t(-1-\gamma)}.$$

Therefore the number of such success is $\geq M_1$ with probability $\leq O(n^{-r-2})$, for some constant M_1. The so the count on the x variables is $\geq M_1$ for at least one a and one z with probability $\leq O(n^{-r})$. We conclude that $c_{\max} \leq M$ with $M = 2M_1^2$ with probability $1 - O(n^{-r})$.

□

Now let $\epsilon, M > 0$ be fixed (as well as an even positive k), and let $\mathcal{E}$ be the event that G_F has $d_{\max} \leq 2n^{1-2\gamma}(1+\epsilon)$, $d_{\min} \geq 2n^{1-2\gamma}(1-\epsilon)$, and $c_{\max} \leq M$. Let $\chi_{\mathcal{E}}$ be the characteristic function of $\mathcal{E}$, i.e. 1 or 0 according to whether or not $\mathcal{E}$ holds. Our main theorem on the trace is the following.

Theorem 1.3 *For all $k \geq 2$ we have that*

$$\mathrm{E}\Big(\chi_{\mathcal{E}}\mathrm{Trace}(A^k)\Big) \leq \left(2n^{1-2\gamma}\right)^k + f(\epsilon, M, k)n^{k(1-2\gamma)}n^{-\min(1,(k/2)(1-2\gamma))},$$

where E *denotes expected value and where f is some function.*

Proof Let $\vec{a}, \vec{b}, \vec{z}, \vec{e}$ be as before with $a_0 = a_k$ and $b_0 = b_k$. Any $\vec{z}$ falls into precisely one of the three following cases: (1) *distinct*, meaning that all z_i are distinct, (2) *quasi-distinct*, meaning that at least one z_i is distinct, but some z_i's are repeated, or (3) *duplicated*, meaning that no z_i's are distinct. If $\vec{z}$ is distinct, then clearly

$$\mathrm{E}\Big(\mathrm{walk}(\vec{a}, \vec{b}, \vec{z}, \vec{e})\Big) = n^{(-1-\gamma)2k}.$$

Hence

$$\sum_{\substack{\vec{a},\vec{b},\vec{z},\vec{e} \\ a_0=a_k, b_0=b_k \\ \vec{z} \text{ distinct}}} \mathrm{walk}(\vec{a}, \vec{b}, \vec{z}, \vec{e}) \leq n^{3k}2^k n^{(-1-\gamma)2k} = \left(2n^{1-2\gamma}\right)^k.$$

Next consider a quasi-distinct $\vec{z}$. Consider the case where z_k is distinct. Fix z_k, a_0, and b_0. Since z_k is distinct, the random variable

$$(a_{i-1}, b_{i-1}) \xrightarrow{(z_i,e_i)} (a_i, b_i)$$

with $i = k$ is 1 with probability $n^{-2-2\gamma}$ independently of all the other random variables with $i \neq k$. Now we let all edges in G_F of colour not z_k be determined, and view the edges coloured z_k as (still undetermined) random variables. We have

$$\sum_{\substack{\vec{a},\vec{b},\vec{z},\vec{e} \\ a_0=a_k, b_0=b_k, z_k \text{ fixed} \\ \vec{z} \text{ with } z_k \text{ distinct}}} \mathrm{E}\Big(\mathrm{walk}(\vec{a}, \vec{b}, \vec{z}, \vec{e})\Big) \leq \mathcal{N}(a_0, b_0, z_k) n^{-2-2\gamma},$$

where $\mathcal{N}(a_0, b_0, z_k)$ denotes the number of walks in G_F of length $k-1$ that do not involve edges of colour z_k but that do have at least one repeated colour. We conclude that

$$\sum \mathrm{E}\Big(\chi_{\mathcal{E}} \mathrm{walk}(\vec{a}, \vec{b}, \vec{z}, \vec{e})\Big)$$

summing over $\vec{a}, \vec{b}, \vec{z}, \vec{e}$ is bounded by $\mathcal{N}_{\max} n^{-2-2\gamma}$, where $\mathcal{N}_{\max}$ is the maximum value of $\mathcal{N}(a_0, b_0, z_k)$ over the event $\mathcal{E}$. To estimate $\mathcal{N}_{\max}$ we note that a vertex will have degree at most $d_{\max}$. The repeated colour occurs for the first two times in $\leq \binom{k}{2}$ positions (in the order $z_1, \ldots, z_k$), and once these positions are fixed then the second time the colour occurs there are at most $c_{\max}$ choices for possible edges. It follows that

$$\mathcal{N}_{\max} \leq (d_{\max})^{k-2} c_{\max} \leq \binom{k}{2} \Big(2(1+\epsilon)n^{1-2\gamma}\Big)^{k-2} M n^{-2-2\gamma}.$$

Summing over (a_0, b_0) we conclude

$$\sum_{\substack{\vec{a},\vec{b},\vec{z},\vec{e} \\ a_0=a_k, b_0=b_k \\ z_k \text{ fixed} \\ \vec{z} \text{ with } z_k \text{ distinct}}} \mathrm{E}\Big(\chi_{\mathcal{E}} \mathrm{walk}(\vec{a}, \vec{b}, \vec{z}, \vec{e})\Big) \leq n^2 \binom{k}{2} \Big(2(1+\epsilon)n^{1-2\gamma}\Big)^{k-2} M n^{-2-2\gamma}.$$

For z_k not distinct but $\vec{z}$ quasi-distinct, consider that $\vec{a}, \vec{b}, \vec{z}, \vec{e}$ represent the existence of a closed walk. So the desired walk exists iff any cyclic rotation of $\vec{a}, \vec{b}, \vec{z}, \vec{e}$ has the same. So one of k cyclic rotations rotates $\vec{z}$ to have z_k distinct, and z_k is chosen from one of n variables, and hence

$$\sum_{\substack{\vec{a},\vec{b},\vec{z},\vec{e} \\ a_0=a_k, b_0=b_k \\ \vec{z} \text{ quasi-distinct}}} \mathrm{E}\Big(\chi_{\mathcal{E}} \mathrm{walk}(\vec{a}, \vec{b}, \vec{z}, \vec{e})\Big) \leq nkn^2 \binom{k}{2} \Big(2(1+\epsilon)n^{1-2\gamma}\Big)^{k-2} Mn^{-2-2\gamma}$$

$$= f_1(k, \epsilon, M) n^{3+(1-2\gamma)(k-2)-2-2\gamma} = f_1(k, \epsilon, M) n^{(1-2\gamma)k-1}.$$

Now we turn to duplicated $\vec{z}$. Our argument is very similar (and a bit easier) than it is in the quasi-distinct case. A duplicated $\vec{z}$ has $\leq k/2$ colours occuring for the first time somewhere in $z_1, \ldots, z_k$, and $\geq k/2$ colours that are repeats. It follows that the total number of loops of length k involving duplicated $\vec{z}$ is therefore

$$\leq f_2(k) d_{\max}^{k/2} c_{\max}^k \leq f_3(k, \epsilon, M) n^{(1-2\gamma)k/2}.$$

Summing over our three estimates for the three types of $\vec{z}$ yields the theorem.

$\square$

Corollary 1.4 *For any $\epsilon, \nu > 0$ the probability that G_F is ν-separated and ϵ-balanced is at least $f(\epsilon, \nu, n)$, where for fixed ϵ, ν and $n \to \infty$ we have*

$$f(\epsilon, \nu, n) = 1 - \Big(1 - (1-\epsilon)^k\Big)(1+\epsilon)^k \nu^{-k} + o(1).$$

Proof For fixed $\epsilon > 0$, fixed positive even k, and a fixed sufficiently large M we have that the event $\mathcal{E}$ occurs with probability $1 - o(1)$, according to Corollary 8. If ρ denotes the maximum of $|\lambda_i|$ for $i > 1$, we have (using the fact that $\lambda_1 \geq 2n^{1-2\gamma}(1-\epsilon)$)

$$\mathrm{E}\Big(\chi_{\mathcal{E}} \rho^k\Big) \leq \mathrm{E}\Big(\chi_{\mathcal{E}} \big(\mathrm{Trace}(A^k) - \lambda_1^k\big)\Big) \leq \Big(2n^{1-2\gamma}\Big)^k \Big(1 - (1-\epsilon)^k + o(1)\Big).$$

It follows that the probability that $\mathcal{E}$ occurs and $\rho \geq (2n^{1-2\gamma})s$ occurs is at most $g(\epsilon, s, n)$, where

$$g(\epsilon, s, n) = 1 - \Big(1 - (1-\epsilon)^k\Big) s^{-k} + o(1).$$

Now choose $s = \nu/(1-\epsilon)$. Then G_F will be ν-separated if $\rho \leq (2n^{1-2\gamma})s$

□

Corollary 1.5 *We have that a random G_F has an independent set of size $\geq n^2/4$ with probability $\leq o(1)$.*

Proof Fix a $\nu > 0$ sufficiently small so that in Theorem 1.1 for small $\epsilon > 0$ we have $f(\nu, \epsilon) < 1/20$. Then for any small ϵ the above corollary (along with Theorem 1.1) implies that an independent set occurs with size $n^2/4$ with probability

$$\leq (1-\epsilon)^k (1+\epsilon)^k \nu^{-k} + o(1).$$

Hence for any ϵ there is an n_0 such that this probability is less than $h(\epsilon)$ for all $n \geq n_0 = n_0(\epsilon)$, where $h(\epsilon) \to 0$ as $\epsilon \to 0$. In other words, for any $\delta > 0$ there is an $\epsilon > 0$ with $h(\epsilon) < \delta$, and hence an idependent set of size $n^2/4$ occurs with probability $\leq \delta$ for $n \geq n_0(\epsilon)$. This is just to say that the aforementioned probability is $\leq o(1)$.

□