Some Graphs with Small Second Eigenvalue

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1 The Second Eigenvalue of a Hypergraph

In this section we will first define the second eigenvalue of 3-regular hypergraph, and then discuss the general notion, as it applies to other regular hypergraphs and graphs. To motivate our definition, notice that if G is an undirected d-regular graph, i.e. each vertex has degree d, then the second largest eigenvalue in absolute value, λ_2, of G's adjacency matrix, A, satisfies

$$|\lambda_2| = \left\| A - \frac{d}{n} \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix} \right\|_{L_2(V)},$$

where $n = |V|$.

Let $G = (V, E)$ be a 3-regular hypergraph; i.e. E is a subset of subsets of V of size 3. We consider the space, $L^2(V)$, of real valued functions on V with the usual inner product; let $e_1, \ldots, e_n$ be the standard basis for $L^2(V)$, where e_i takes the value 1 on the i-th vertex of V and 0 elsewhere. It is natural to construct from G a trilinear form, that is a map from τ, mapping triples of vectors in $L^2(V)$ to $\mathbf{R}$, namely

$$\tau\left(\sum_{i=1}^n \alpha_i e_i, \sum_{j=1}^n \beta_j e_j, \sum_{k=1}^n \gamma_k e_k\right) = \sum_{i,j,k} \alpha_i \beta_j \gamma_k \tau_{i,j,k}$$

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where

$$\tau_{i,j,k} = \begin{cases} 1 & \text{if } \{i,j,k\} \in E \\ 0 & \text{otherwise} \end{cases}$$

Let $\vec{1}$ denote the all 1's vector, and let $\vec{1} \otimes \vec{1} \otimes \vec{1}$ denote the all 1's trilinear form (i.e. the form ν defined as τ but with $\nu_{i,j,k} = 1$ for all i, j, k). We define τ to be d-degree-regular if

$$\sigma = \tau - \frac{d}{n} \vec{1} \otimes \vec{1} \otimes \vec{1}$$

has the property that

$$\sigma(\vec{1}, u, v) = \sigma(u, \vec{1}, v) = \sigma(u, v, \vec{1}) = 0$$

for all u and v. In this case we define the *second eigenvalue* of G (and τ) to be

$$\lambda_2 = \|\sigma\|_{L^2(V)} = \sup_{\|u\|=\|v\|=\|w\|=1} |\sigma(u, v, w)|.$$

We pause for a few remarks. First of all, for simplicity we have abused some of the usual tensor conventions; in particular, we will often identify spaces and elements of spaces with their duals implicitly when the distinction serves no purpose and/or needlessly complicates the notation or discussion. For example, the all 1's trilinear form is really the dual of $\vec{1} \otimes \vec{1} \otimes \vec{1}$ in the usual conventions.

Second, just as ordinary directed graphs can have multiple edges and self-loops, we can accomodate such notions here. If an edge $\{i, j, k\}$ occurs with multiplicity m, then the six corresponding entries of τ will have an m instead of a 1. Also, an edge which is a self-loop, $\{i, i, i\}$, should contribute 6 to $\tau_{i,i,i}$, on the principle that the contribution should be determined so that each edge contributes a total of 6 to all the entries of τ. Similarly an edge, $\{i, i, j\}$ should contribute 2 to each of $\tau_{i,i,j}$, $\tau_{i,j,i}$, and $\tau_{j,i,i}$.

Thirdly, we can also handle the notion of directed edges. We say that a directed 3-regular hypergraph is a hypergraph where each edge $\{i, j, k\}$ has a specified order; equivilently, E is a subset of $V \times V \times V$. Now the trilinear form τ is defined by $\tau_{i,j,k}$ being the multiplicity of the edge (i, j, k). From a directed hypergraph we can form an undirected hypergraph, i.e. the usual notion of hypergraph, by forgetting about the ordering on the edges. The new trilinear form is the symmetrized version of the old one. This also explains why undirected degenerate edges such as $\{i, i, i\}$ should contribute 6 to $\tau_{i,i,i}$ in the usual (undirected) notion of hypergraph. Identifying a hypergraph with its trilinear form, we can think of undirected hypergraphs as special cases of directed hypergraphs.

Fourth, we define for any trilinear form on $L^2(V)$, μ, its first eigenvalue to be its norm with respect to $L^2(V)$, i.e.

$$\lambda_1(\mu) = \|\mu\|_{L^2(V)} = \sup_{\|u\|=\|v\|=\|w\|=1} |\mu(u,v,w)|.$$

If u, v, w is a triple achieving the above sup, we shall call (u, v, w) or $u \times v \times w$ an eigenvector. We can also define d-degree-regular for a general trilinear form as above, and therefore extend the notion of second eigenvalue to directed hypergraphs. It is easy to check that an undirected hypergraph is d-degree-regular if for every vertices i, j, there are exactly d vertices k for which $\{i, j, k\}$ is an edge (with the case $i = j$ included); similarly, a directed hypergraph is d-degree-regular if for every vertices i, j there are d edges of each of the forms $(i, j, \cdot)$, $(i, \cdot, j)$, and $(\cdot, i, j)$.

Fifth, it becomes clear how we want to define the second eigenvalue in the more general case. For any sets, $V_1, \ldots, V_k$, we define the first eigenvalue of a k-linear form mapping $V_1 \times \cdots \times V_k$ to $\mathbf{R}$ as its norm with respect to the norms $L^2(V_i)$. A k-linear form, τ, is d-degree-regular if

$$\sigma = \tau - \frac{d}{n}\vec{1}_{V_1} \otimes \cdots \otimes \vec{1}_{V_k},$$

with n being the largest among the $|V_i|$'s, satisfies

$$\sigma(v_1, \ldots, v_k) = 0$$

whenever some v_i is $\vec{1}_{V_i}$. In this case we define the second eigenvalue of τ to be $\lambda_1(\sigma)$. Notice that for a directed graph, with adjacency matrix A, which is d-degree-regular, it's second eigenvalue in the new sense is the square root of the (classical) second largest eigenvalue of AA^T.

Sixth, the first and second eigenvalue of multilinear forms as defined above are always non-negative numbers. While much of the literature on graphs considers the second eigenvalue as the eigenvalue with the second largest absolute value, sometimes one simply considers the second largest (positive) eigenvalue. Here the distinction is lost.

Lastly, we could define the eigenvalues and/or norms of multilinear forms with respect to other norms on the space of functions on V, such as $L^p(V)$ for any $1 \le p \le \infty$. In the applications we have in mind, it does not seem to help to consider the other norms; and, in fact, one can say more about the $L^2(V)$ norms than for other values of p.

We return to the discussion of 3-regular hypergraphs. There is a lot of structure for graphs that we don't know how to carry over to 3-regular hypergraphs. For example, we don't know how multiply two hypergraphs. However, a fair amount of the theory for graphs can be generalized to 3-regular hypergraphs. We will prove:

Theorem 1.1 *The first eigenvalue of a d-degree-regular 3-regular hypergraph is $dn^{1/2}$; i.e. the associated trilinear form τ is maximized on $(\vec{1}, \vec{1}, \vec{1})$.*

Theorem 1.2 *A symmetric trilinear form τ has a unit vector v such that $\|\tau\| = |\tau(v, v, v)|$.*

Theorem 1.3 *Every 3-regular, d-degree-regular hypergraph has second eigenvalue $\geq \sqrt{d}$.*

Theorem 1.4 *For any d and n, a random d-degree-regular 3-regular hypergraph on n vertices has second eigenvalue $\leq C\sqrt{d}$ for some absolute constant C.*

All of the above theorems have generalizations to r-regular hypergraphs. The last theorem is quite interesting, in that implies the existence of hypergraphs with such low second eigenvalues that they could significantly improve on a class of applications of graphs with low second eigenvalues. Unfortunately we do not have an explicit construction for such hypergraphs. Most explicit constructions of graphs with small second eigenvalues are Cayley graphs of some sort. There is a natural generalization of these graphs, which we call "Cayley hypergraphs," but it turns out that these hypergraphs can never have such low second eigenvalues as those of theorem 1.4. We will discuss Cayley hypergraphs in section mumble. We also discuss second eigenvalue for bipartite graphs, but this does not yield anything new applications.

2 Basic Facts about Hypergraphs

In this section we will prove some basic facts about the eigenvalues of hypergraphs. We will call a multilinear form, τ, d-degree-regular if it satisfies the conditions mumble.

Theorem 2.1 *Let τ be a non-negative, d-degree-regular k-linear form on n vertices. Then the first eigenvalue of H is $dn^{(k-2)/2}$, with $\vec{1} \otimes \cdots \otimes \vec{1}$ being an eigenvector.*

Proof We shall proceed by induction on k. For $k = 1$ this is clear, for then the associated multilinear form, τ, is just a one-dimensional vector proportional to $\vec{1}$. For general k, let the associated trilinear form, τ, take its maximum over the product of unit balls at $(u_1, \ldots, u_k)$. Since τ has non-negative coefficients, we can assume that each u_i has non-negative coefficients (or replace the u_i by such vectors while preserving their norm and without decreasing the absolute value of τ at $(u_1 \ldots, u_k)$). But viewing u_1 as fixed, the $k-1$ form $\tau(u_1, \cdot, \ldots, \cdot)$ is easily seen to be a non-negative, θ-degree-regular form where θ is the sum of the components of u_1; therefore, by induction, then norm of this $k-1$-linear form is $\theta n^{(k-3)/2}$, with $\vec{1} \otimes \cdots \otimes \vec{1}$ being a coresponding eigenvector. But the ratio of θ to $\|u_1\|$ is maximized when u_1 is proportional to $\vec{1}$, and so the norm of τ is $dn^{(k-2)/2}$, with $\vec{1} \otimes \cdots \otimes \vec{1}$ being a corresponding eigenvector.

□

Theorem 2.2 *Let τ be a symmetric k-linear form. Then there is a vector $u \in L^2(V)$ such that $u \otimes \cdots \otimes u$ is a first eigenvector for τ.*

Lemma 2.3 *Let ν be a symmetric 2-linear form. If u and v are unit vectors such that $\|\nu\| = |\nu(u, v)|$, then also $\|\nu\| = |\nu(u, u)|$.*

Proof By linear algebra, ν is diagonalizable with an orthonormal basis. Let $\widetilde{\nu}$ be the associated endomorphisn of $L^2(V)$. If $\widetilde{\nu}$'s largest eigenvalue in absolute value is λ, then $\|\nu\| = |\lambda| = \|\widetilde{\nu}\|$. Since

$$|\lambda| = |\nu(u, v)| = |(\widetilde{\nu}(u), v)| \leq \|\widetilde{\nu}(u)\| \|v\| = \|\widetilde{\nu}(u)\|,$$

u must be an eigenvector with corresponding eigenvalue $= \pm\|\nu\|$, and so $\|\nu\| = |\nu(u, u)|$.

□

The theorem easily follows. If $u_1 \otimes \cdots \otimes u_k$ is a first eigenvector for τ, then applying the lemma to the bilinear form $\tau(\cdot, \cdot, u_3, \ldots, u_k)$ we see that we can replace u_2 by u_1. Similarly we can replace all the u_i by u_1, successively, proving the theorem.

□

We now give a lower bound for the second eigenvalue. Since the projection of a vector with a ones and $n - a$ zeros onto $\vec{1}^{\perp}$ has norm $\sqrt{a(n-a)/n}$, the following proposition follows immediately from the definition of second eigenvalue:

Proposition 2.4 *Let H be a k-regular, d-degree-regular hypergraph. For any subsets $U_1, \ldots U_k$ of V, the number of edges in $U_1 \times \cdots \times U_k$ is*

$$\frac{d}{n}|U_1| \cdots |U_k| + \theta\lambda_2 \sqrt{\frac{|U_1|(n - |U_1|)}{n}} \cdots \sqrt{\frac{|U_k|(n - |U_k|)}{n}}$$

for some θ with $|\theta| \leq 1$, and where λ_2 is the second eigenvalue of the hypergraph.

Now take any $U_1, \ldots, U_{k-1}$ consisting of one vertex. Then we can find a U_k of size $n - d$ for which E contains no edges in $U_1 \times \cdots \times U_k$. It follows that:

Proposition 2.5 *Let H be as in proposition 2.4. Then*

$$\lambda_2 \geq \sqrt{\frac{d(n-d)}{n}}.$$

In the literature there are many methods for obtaining upper and lower bounds on the second eigenvalue of graphs. We remark that when we consider the bilinear form $\tau(v_1, \ldots, v_{k-2}, \cdot, \cdot)$ with each v_i being one of the standard vectors, e_j, we get a d-regular graph, and can therefore apply lower bounds known for the second eigenvalue of graphs. The preceding proposition is an example. There are two other methods for obtaining lower bounds, and they yield:

Proposition 2.6 *Mumble.*

Proposition 2.7 *Mumble.*

As for obtaining upper bounds, the strongest methods don't seem to directly generalize. The so-called "trace method," which uses multiplication of graphs and taking the trace of a graph (its adjacency matrix), does not have an obvious generalization (that we see). However, for graphs which are Cayley graphs, such as those of [LPS86] and [Mar87], there is a natural generalization, and it turn out that to analyze their second eigenvalue is suffices to analyze the second eigenvalue of the corresponding graph (and to understand the representations of the underlying group), so the trace method can be applied there. For random graphs, we don't know how to apply results that use the trace method, such as those of [Wig55], [**?**], [FW]. However the method of Kahn and Szemeredi, in [KS], does generalize quite readily, and we can prove:

Theorem 2.8 *Mumble*

Proof Mumble.

For applications we would like to come up with constructions to match this bound. Often we would like the graph to be constructible in poly log of the number of nodes, for example in using weak random sources. We do not have an explicit construction to match the bound in theorem 1.4, not even one which is constructable in polynomial time in the number of nodes. We will describe two types of constructions; one is very easy, and yields a weak eigenvalue bound. The second type is an analogue of Cayley graphs, which we call Cayley hypergraphs. This gives somewhat improved bounds, but we also give a lower bound to show that Cayley hypergraphs cannot match the bound of theorem 1.4. We find this interesting because most explicit constructions of graphs with small second eigenvalue that we know are Cayley graphs. Perhaps there is a better generalization of this concept that can yield hypergraphs with small second eigenvalue. We finish this section by giving an easy construction for a graph with slightly small second eigenvalue.

Let $G = (V, E)$ be any graph which is d-regular, and let $\sigma_1, \ldots, \sigma_n$ be any set of n permutations on the numbers $\{1, \ldots, n\}$ with the property that for any i,

$$\{\sigma_1(i), \ldots, \sigma_n(i)\} = \{1, \ldots, n\}.$$

Let $\widetilde{G} = (V, \widetilde{E})$ we the 3-regular hypergraph with edge set

$$\widetilde{E} = \{(i, j, k) \mid (j, \sigma_i(k)) \in E\}.$$

Proposition 2.9 *If λ_2 is the second eigenvalue of G, then the second eigenvalue of $\widehat{G}$ is no greater than $\sqrt{n}|\lambda_2|$.*

Proof Mumble.

3 Cayley Hypergraphs

Let G be a group, and H a subset of G. The *Cayley graph* on G generated by H is defined to be the graph with vertex set G and edge set

$$\{(x, y) \mid xy^{-1} \in H\}.$$

(We do not require that H generate G, nor that $H = H^{-1}$.) This gives a d-regular directed graph, and if $H = H^{-1}$ we can view the graph as undirected.

We define the *Cayley sum graph* similarly, though taking the edge set to be

$$\{(x, y) \mid xy \in H\}.$$

Perhaps the easiest way to get a 3-regular hypergraph from this data is to keep G as the set of vertices and to take

$$\{(x, y, z) \mid xyz \in H\}$$

as the edge set; we will call this hypergraph the *(3-regular) Cayley hypergraph* on G and H.

The eigenvalues of Cayley graphs and, as we shall see, hypergraphs, can often be estimated when one understands the decomposition of $L^2(G)$ under the right regular representation. We recall the following facts about representations of finite groups (see [**?**] for details). We will, for the moment, let $L^2(G)$ denote the space of complex-valued functions on G with the usual inner product:

$$(u, v) = \sum_{g \in G} u_g \overline{v_g}.$$

$L^2(G)$ can be decomposed into subspaces

$$L_2(G) = \bigoplus_{i=1}^{r} E_i$$

with the following conditions:

1. Each E_i is invariant under the natural action of G on $L_2(G)$, given by $g(u(x)) \equiv u(gx)$.
2. $\dim(E_i) = d_i^2$ for some d_i corresponding to the dimension of an irreducible unitary representation of G, $\rho_i : G \to \mathrm{Gl}(\mathbf{C}, d_i)$, in the sense that a complete orthogonal basis for E_i is given by the d_i^2 entries of ρ_i with respect to any basis of $\mathbf{C}^n$. Also, the norm of each coefficient of ρ_i, as an element of $L^2(G)$, has norm $\sqrt{n/d_i}$.
3. r is equal to the number of conjugacy classes in G.

It follows that the matrix A of any Cayley graph on G vanishes outside the $E_i \times E_i$ blocks, in the sense that $\tau(u, v) = 0$ if u and v are contained in different E_i's. More generally, the t-regular hypergraph generated by G and H vanishes outside the $E_i \times \cdots \times E_i$ blocks. Let us assume, for simplicity, that the Cayley graph is generated by an H which satisfies $H{=}H^{-1}$. It then follows that the eigenvalues of A are real, and there is an orthogonal set of real eigenvectors.

Theorem 3.1 *Let the eigenvalues of A restricted to E_i be $\lambda_1, \dots \lambda_{d_i^2}$. Then the norm of the t-linear form associated to the t-regular Cayley hypergraph on G and H is*

$$\left(\frac{n}{d_i}\right)^{(t-2)/2} \max_j |\lambda_j|.$$

Proof For simplicity we will first prove this for 3-regular hypergraphs, and then indicate how to generalize the proof. Let A and τ, by abuse of notation, denote the adjacency matrix and bilinear form, respectively, associated to the Cayley graph restricted to the subspace E, corresponding to the d dimensional representation ρ. For any basis of $\mathbf{C}^d$, we can consider the coefficients of ρ, $\{\rho_{i,j}\}$, with respect to a given basis for E, $w_1, \dots, w_d$. For any i, j, k, l, we have

$$\sum_{xy=h} \rho_{i,j}(x)\rho_{k,l}(y) = \sum_{g \in G} \rho_{i,j}(g)\rho_{k,l}(g^{-1}h)$$

which, using the fact that ρ is a unitary representation, is

$$\sum g \in G \sum_{m=1}^{d} \rho_{i,j}(g)\overline{\rho_{l,m}(g)}\rho_{m,k}(h) = \delta_{i,k}\left(\frac{n}{d}\right)\rho_{j,l}(h),$$

where $\delta_{i,k}$ is the Kronecker delta function. It follows that τ is given by

$$\tau\left(\sum_{i,j} \alpha_{i,j}\rho_{i,j}, \sum k, l\beta_{k,l}\rho_{k,l}\right) = \sum_{i,j,k,l} \alpha_{i,j}\beta_{k,l}\delta_{i,k}M_{j,l} = \sum_{i,j,l} \alpha_{i,j}\beta_{i,l}M_{j,l}$$

where $M_{j,l}$ is given by the matrix equation

$$M = \left(\frac{n}{d}\right)\sum_{h \in H} \rho(h).$$

Since $H = H^{-1}$, it follows that M is a real symmetric matrix, and therefore is diagonalizable by a set of real eigenvectors in $\mathbf{R}^d$ with real eigenvalues. Take these eigenvectors as the basis of $\mathbf{C}^d$, $w_1, \dots, w_d$. Then M becomes a diagonal matrix. Since the norm of each $\rho_{i,j}$ is $\sqrt{n/d}$, it follows that for each i, j, the function $\rho_{i,j}$ is an eigenvector of the adjacency matrix, with eigenvalue 0 if $i \neq j$, and eigenvalue $\lambda_{i,i} = \frac{d}{n}M_{i,i}$ if $i = j$.

Now let ν be the trilinear form associated to the 3-regular hypergraph generated by G and H. A similar calculation shows that ν is given by

$$\nu\left(\sum_{i,j} \alpha_{i,j}\rho_{i,j}, \sum_{k,l} \beta_{k,l}\rho_{k,l}, \sum_{q,r} \gamma_{q,r}\rho_{q,r}\right) = \sum \alpha_{i,j}\beta_{k,l}\gamma_{q,r}\delta_{i,k}\delta_{j,q}\delta_{l,r}\frac{n}{d}M_{l,l}$$

$$= \sum \alpha_{i,j}\beta_{i,l}\gamma_{j,l}\frac{n}{d}M_{l,l}$$

It follows that $\nu(\rho_{i,i}, \rho_{i,i}, \rho_{i,i})$ is $\frac{n}{d}M_{i,i}$, and so

$$\|\nu\| \geq \sqrt{\frac{n}{d}}\lambda_{i,i}$$

for each i. We claim this hitting the ν with $\rho_{i,i}$ is the best that we can do, i.e.

Proposition 3.2 *We have*

$$\|\nu\| = \max_i \sqrt{\frac{n}{d}}|\lambda_{i,i}|.$$

Proof Let $\alpha_{i,j}, \beta_{k,l}, \gamma_{q,r}$ be given with

$$\sum_{i,j} \alpha_{i,j}^2 = \sum_{i,j} \beta_{i,j}^2 = \sum_{i,j} \gamma_{i,j}^2 = 1.$$

It suffices to obtain the estimate

$$\sum_{i,j,k} \alpha_{i,j}\beta_{i,k}\gamma_{j,k} \leq 1.$$

Letting

$$a_{j,k} = \sum_i \alpha_{i,j}\beta_{i,k},$$

we see that applying Cauchy-Schwartz inequality yields

$$\sum_{j,k} a_{j,k}^2 \leq \sum_{j,k}\left(\sum_i \alpha_{i,j}^2\right)\left(\sum_i \beta_{i,k}^2\right) = 1$$

and so

$$\sum_{j,k} a_{j,k}\gamma_{j,k} \leq \left(\sum_{j,k} a_{j,k}^2\right)^{1/2}\left(\sum_{j,k} \gamma_{j,k}^2\right)^{1/2} \leq 1.$$

□

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