Cohomology in Grothendieck Topologies and Lower Bounds in Boolean Complexity

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Abstract

This paper is motivated by questions such as P vs. NP and other questions in Boolean complexity theory. We describe an approach to attacking such questions with cohomology, and we show that using Grothendieck topologies and other ideas from the Grothendieck school gives new hope for such an attack.

We focus on circuit depth complexity, and consider only finite topological spaces or Grothendieck topologies based on finite categories; as such, we do not use algebraic geometry or manifolds.

Given two sheaves on a Grothendieck topology, their *cohomological complexity* is the sum of the dimensions of their Ext groups. We seek to model the depth complexity of Boolean functions by the cohomological complexity of sheaves on a Grothendieck topology. We propose that the logical AND of two Boolean functions will have its corresponding cohomological complexity bounded in terms of those of the two functions using "virtual zero extensions." We propose that the logical negation of a function will have its corresponding cohomological complexity equal to that of the original function using duality theory. We explain these approaches and show that they are stable under pullbacks and base change. It is the subject of ongoing work to achieve AND and negation bounds simultaneously in a way that yields an interesting depth lower bound.

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1 Introduction

Over twenty years ago lower bounds for algebraic decision trees were obtained by counting connected components (and in principle the sum of the Betti numbers) of associated topological spaces (see [DL76, SY82, BO83]). This lead to a hope that problems such as P vs. NP, viewed as lower bound problems in Boolean circuit complexity of a Boolean function, could be studied via cohomology, e.g., the sum of the Betti numbers of a topological space (associated in some way to the function). We are unaware of any essential progress in this direction to date. (But see [Sma87] for a success of algebraic topology and the braid group in another notion of complexity.) In fact, there are what might be called "standard obstacles" to this topological approach in Boolean complexity. In this paper we show that two obstacles in depth complexity can be circumvented in a natural way provided that we (1) generalize the notion of Betti number using sheaf theory and the derived category, and (2) replace topological spaces with Grothendieck topologies[1], see [SGA4][2]. We explain our approach to this circumvention, and give some foundational theorems that we hope will be useful in our ongoing work of seeking (Grothendieck) topological models for Boolean functions to yield interesting lower bounds in Boolean depth and, perhaps later, size complexity.

There is a lot of appeal to trying to model Boolean complexity via cohomology (e.g., the sum of the Betti numbers) over the appropriate space or topology. First, cohomology is a natural invariant of spaces for which there is and wealth of intuition, examples, and tools, some quite sophisticated. Second, cohomology often takes infinite or large dimensional vector spaces and extracts more concise and meaningful information. Third, cohomology has much overlap with and applications to combinatorics; toric varieties is one example; more basically, the inclusion/exclusion principle follows from, via the standard resolution, the fact that the n-simplex (what we below call Δ_n) has the Betti numbers of a point; so the general study of cohomology (especially when higher cohomology groups don't vanish) can be viewed as a vast

[1]A Grothendieck topology (by which we mean a "*site*," as in [SGA4.II.1.1.5]) is a generalization of a topological space; a Grothendieck topology has just enough structure to define sheaf theory and therefore, cohomology, and has the properties that, roughly speaking, (1) an "open set" can be "included" in another (or itself) in more than one way, and (2) the notion of "local" or "refinement" is not necessarily the "canonical" one.

[2]Throughout this paper, we use notation such as [SGA4.V.2.3.6] to refer to SGA4 (i.e., [sga72a, sga72b, sga73]), *exposé* V, *section* (or, in this case, *exercice*) 2.3.6.

generalization of inclusion/exclusion. Fourth, sheaf cohomology and many of its tools makes sense over an arbitrary topological space or Grothendieck topology, so there are many possibilities for modelling Boolean functions by sheaves on Grothendieck topologies.

We emphasize, regarding the fourth point, that our work here does not involve algebraic geometry or analysis (e.g., bounds based on degrees and intersection theory, morse theory, such as used in [SY82, BO83]). Here we will consider only finite topological spaces and, more generally, Grothendieck topologies whose underlying category is finite (and "semitopological" as defined below). Such spaces can loosely model some aspects of smooth manifolds while being, in a sense, not as restricted (or rich in structure) as manifolds or schemes. On the other hand, as shown in [SGA4.I–VI], any Grothendieck topology has analogues of sheaves, cohomology, and related concepts that are strikingly similar to what one is accustomed to from areas such as analysis, algebraic geometry, group theory, etc.

Let us begin by describing two obstacles to modelling depth complexity with cohomology. For any integer $n \geq 1$, consider the Boolean functions on n variables, $f\colon \{0,1\}^n \to \{0,1\}$ (with 1 being TRUE). We define the functions $0, 1, x_1, \neg x_1, x_2, \ldots, \neg x_n$ to be of depth[3] 0, where x_i is the i-th coordinate of $\{0,1\}^n$, and inductively define a function to be of depth i if it equals $f \wedge g$ or $\neg(f \wedge g)$ with f, g of depth $i-1$ (here $\neg f$ is $1-f$, its logical negation, and $f \wedge g$ is fg, their logical conjunction or AND); the depth complexity of f is the smallest depth in which it appears. Let h map Boolean functions to the non-negative reals, with

$$h(f \wedge g) \leq c_1 + c_2 \max\bigl(h(f), h(g)\bigr) \tag{1}$$

and

$$h(\neg f) = h(f), \tag{2}$$

for any Boolean functions f, g; then it is easy to see that

$$\text{depth complexity } (f) \geq \log_{c_2} \frac{h(f)}{M + \frac{c_1}{c_2 - 1}},$$

where M is the maximum of $h(x_1), h(\neg x_1), \ldots, h(\neg x_n)$; such an h is an example of what is called a *formal complexity measure* (see [Weg87]). Assume that to each Boolean function, f, we have associated a topological space, U_f,

[3]Morally the functions $0, 1$ should probably be defined also to be of depth -1.

and we set $h(f)$ to be the sum of the Betti numbers of U_f. (Note that we will soon broaden our class of h's, but Betti numbers of topological spaces is a good place to start.) We wish to verify equations (1) and (2) for some c_1, c_2, c_3.

The first obstacle is that U, V in the plane can be each diffeomorphic to an open disk, while $U \cap V$ and $U \cup V$ each have an arbitrarily high sum of Betti numbers (make their boundaries intersect "wavily"). This means that if $U_{f \wedge g}$ is $U_f \cap U_g$ or $U_f \cup U_g$, we do not anticipate that a general principle will establish equation (1). In terms of sheaf theory, the problem is that the relation between the Betti numbers follows from the short exact sequence

$$0 \to \mathbb{Q}_{U_f \cap U_g} \to \mathbb{Q}_{U_f} \oplus \mathbb{Q}_{U_g} \to \mathbb{Q}_{U_f \cup U_g} \to 0,$$

where $\mathbb{Q}_A$ denotes the rationals, $\mathbb{Q}$, restricted to A and extended by zero elsewhere; since $\mathbb{Q}_{U_f}, \mathbb{Q}_{U_g}$ control only one (nonzero) term in the sequence, we have no control on the Betti numbers arising from the other two. Here we will propose a general principle from which U_f and U_g will control the sequence, providing we pass to Grothendieck topologies with "enough virtual zero extensions" or "no composition conflicts." Once we make these ideas precise, the proof follows immediately from two short exact sequences. We shall show that Grothendieck topologies with "enough virtual zero extensions" exist; in fact, so many exist that we have only begun to study them at this point. Free categories always have this property, but they are of homological dimension at most one and this may indicate that we should look elsewhere for interesting categories. Pulling back preserves "virtual zero extensions" and can add new ones, and therefore there are other strategies for building interesting categories by successive pullbacks, in particular fiber products.

It would be possible that $U_{f \wedge g}$ has "nothing to do" with U_f and U_g, so that there really is no obstacle as described above. But most approaches we have seen in various types of complexity have some similar relations, and if not then one still needs a method to establish equation (1). Note our approach insists nothing about $U_{f \vee g}$, and we deal with $f \vee g$ indirectly via $f \vee g = \neg((\neg f) \wedge (\neg g))$ (the minimum depth of a function being closed under negation).

The second obstacle is to establish $h(f) = h(\neg f)$; of course, it would suffice to establish $h(\neg f) \leq c_1 + c_2 h(f)$, but in most settings f and $\neg f$ have the exact same complexity. Since the association between f and $\neg f$ is so natural and simple, we insist, in this paper, that $h(f) = h(\neg f)$. Again, it is possible that the models for f and $\neg f$ have "nothing to do" with each other, and that $h(f) = h(\neg f)$ by accident or some less direct means. However, we feel that it could be productive to look for a direct reason for $h(f) = h(\neg f)$.

One possibility is that f and $\neg f$ have the same model. This does not work well if for each f, g we have $U_{f\wedge g} = U_f \cap U_g$, for then

$$U_f = \bigcap_{\alpha\in\{0,1\}^n,\quad f(\alpha)=0} U_{\chi_\alpha},$$

with χ_α being the characteristic function of α (one at α and zero elsewhere). In particular, for any $\beta \neq \gamma$ we have

$$U_{\chi_\beta} = U_{1-\chi_\beta} = \bigcap_{\alpha\neq\beta} U_{\chi_\alpha} \subset U_{\chi_\gamma}.$$

Similarly $U_{\chi_\gamma} \subset U_{\chi_\beta}$, and U_f is independent of f. Even if we don't have $U_{f\wedge g} = U_f \cap U_g$, it still seems hard to deal with equation (1) assuming f and $\neg f$ have the same model.

Another possibility is that f and $\neg f$ could be different but have the same Betti number sum because of duality between the cohomology groups of f and $\neg f$, which we will soon make precise. Let us mention that this duality is akin to Poincaré or Serre duality. Furthermore, the main technical result in this paper is to define what it means for a morphism of the topologies we study to be "strongly n-dimensional" (this looks like a special type of relative Poincaré duality), and to prove that this property is stable under base change. This means that the pulling back that we were considering to obtain "enough virtual zero extensions" (to satisfy equation (1)) preserves strong dimensionality.

Henceforth we give a more precise description of our approach.

If X is a Grothendieck topology (for example, a topological space), and F, G are sheaves of $\mathbb{Q}$-vector spaces on X, we define the *cohomological complexity* of (X, F, G) as

$$\mathrm{cc}_X(F, G) = \sum_{i\geq 0} \dim_{\mathbb{Q}} \mathrm{Ext}^i_X(F, G).$$

(In particular, $\mathrm{cc}_X(\mathbb{Q}, \mathbb{Q})$ is the sum of the Betti numbers of X.) By a *sheaf model (on n variables)* we mean an association to each Boolean function on

n variables, $f\colon \{0,1\}^n \to \{0,1\}$, a tuple (X_f, F_f, G_f); by the homological complexity of f we mean that of the associated tuple.

In this paper we limit our focus as follows. We will consider only the *depth complexity* of a Boolean function. Furthermore we consider only Grothendieck topologies whose underlying category, $\mathcal{C}$, is *finite*, meaning having finitely many morphisms, and *semitopological*, meaning that the only morphism from an object to itself is the identity morphism; in case any two objects have at most one morphism between them, we shall call $\mathcal{C}$ *of topological type*[4], as the associated topos is equivalent to one of a topological space, using Theorem 2.1. Moreover we consider only the *grossière* topology on the category, $\mathcal{C}$ (in which the sheaves are just the presheaves), since by Theorem 2.1 this essentially loses no generality. The sheaves we will consider will be sheaves of finite dimensional vector spaces over the rationals, $\mathbb{Q}$, denoted $\mathbb{Q}(\mathcal{C})$. The standard resolution (see Section 2.10) then implies that the cohomological complexity is always finite in the above case (i.e., the case of finite dimensional sheaves of $\mathbb{Q}$-vector spaces on finite semitopological categories).

Let us return to our first obstacle. Namely, fix a Grothendieck topology, X, and sheaves F, G for which $\mathrm{cc}(F,G) = 0$. Then if U is an open set and Z its closed complement, the short exact sequence

$$0 \to G_U \to G \to G_Z \to 0$$

shows that $\mathrm{cc}(F, G_U) = \mathrm{cc}(F, G_Z)$. Consider a sheaf model $f \mapsto (X, F, G_{U_f})$ where $f \mapsto U_f$ is an association of an open set, U_f, to each Boolean function, f, with the property that $U_{f\wedge g} = U_f \cap U_g$ for all f, g. Then the homological complexity of $f \wedge g$ is bounded by that of f and g provided that there exists an exact sequence

$$0 \to G_{U_f} \to G_{U_f \cap Z_g} \oplus H \to G_{Z_g} \to 0$$

for some sheaf, H, where Z_g is the closed complement of U_g and where $G_{U_f \cap Z_g} = G \otimes \mathbb{Q}_{U_f} \otimes \mathbb{Q}_{Z_f}$ with $\mathbb{Q}_{U_f}, \mathbb{Q}_{Z_g}$ being the usual open and closed extensions by zero. Such an H will exist only for very special U_f, U_g when $\mathcal{C}$ is topological. However, such H's exist whenever $\mathcal{C}$ has "enough arrows to avoid composition conflicts" (see Section 4).

[4]Note that the term "topological category" has another meaning, namely as a category whose sets of objects and morphisms are topological spaces with source and target maps being continuous maps.

Duality gives a non-trivial equality of cohomological complexity, given by Ext duality, akin to Serre (or Poincaré) duality. This describes situations where

$$\operatorname{cc}(X, F, G) = \operatorname{cc}(X, G, F') \tag{3}$$

for a natural sheaf F' (the star denoting the dual vector space), so that if $F' = F \otimes \omega$ for a vector bundle ω (see Section 2.8), then

$$\operatorname{cc}(X, F, G) = \operatorname{cc}(X, F, G' \otimes \omega^\vee)$$

where $\omega^\vee$ is the dual sheaf (see Section 2.8), provided that G' exists as well. More precisely, we shall define a simple functor $! \to *$ on the derived category $\mathcal{D} = \mathcal{D}^{\mathrm{b}}(\mathbb{Q}(\mathcal{C}))$ (of bounded complexes in $\mathbb{Q}(\mathcal{C})$) such that

$$\operatorname{Hom}_{\mathcal{D}}(F, G) = \bigl(\operatorname{Hom}_{\mathcal{D}}(G, F^{! \to *})\bigr)^* \tag{4}$$

(we write $F^{!\to *}$ for $(! \to *)F$ at times) for all $F, G \in \mathcal{D}$ (in other words, $! \to *$ is the Serre functor, see [BK89, BO01, BLL04] and Section 2.12). When F, G are sheaves and when $F^{!\to *} \simeq F'[n]$ for some F' and n, then equation (3) holds. If $G_0 = G_{U_f}$, then we would hope to show that either $G_{U_{\neg f}}$ or $G_{Z_{\neg f}}$ is $G_1 = G_0' \otimes \omega^\vee$, or $G_2 = G_1' \otimes \omega^\vee$, etc. (or the same with $G_0 = G_{Z_f}$).

Let us specialize our discussion. Consider $F = \mathbb{Q}$, and consider those categories $\mathcal{C}$ for which $\mathbb{Q}^{!\to *} \simeq \mathbb{Q}[n]$ for some n. In this case we say that $\mathcal{C}$ is *strongly n-dimensional*; (Some (but not all) categories arising from coverings of manifolds have this property.) We can use the product and fiber product to construct new such categories out of old ones, but our fiber product constructions sometimes require a relative notion of strong dimensionality. Namely, we say that a functor $f \colon \mathcal{X} \to \mathcal{S}$ is strongly n-dimensional if

$$(! \to *)_{\mathcal{X}} f^* \simeq f^* (! \to *)_{\mathcal{S}}[n] \tag{5}$$

(if $\mathcal{S}$ is a point, then it suffices to test this condition on $\mathbb{Q}$, which amounts to $\mathcal{X}$ being strongly n-dimensional). The most difficult theorem in this paper is to show that strong dimensionality of a morphism is closed under base change (thus giving a fiber product construction). This is proven by giving an equivalent, fiberwise n-dimensionality, which is clearly closed under base change due to its local nature.

It follows that our proposed approaches to equations (1) and (2) are both "compatible" with pulling back or fiber products in some sense. We hope that fiber products and pull backs, combined with a sufficient collection of examples (as we begin to establish in Section 3) will yield interesting sheaf models.

Before describing the rest of this paper, we make two remarks. First, the size of the categories needed to have enough "virtual zero extensions" seems to be (in number of morphisms) doubly or triply exponential in n, meaning that our techniques are not "natural" in the sense of [RR97]. Finally, Mulmuley and Sohoni have an approach to circuit complexity that is very different from ours in that it uses algebraic geometry, in particular geometric invariant theory; see the series of papers beginning with [MS01].

In Section 2 we fix a lot of notation and recall various facts needed later; all facts are known or follow easily from known results. In Section 3 we describe how certain topological spaces (e.g., smooth manifolds) are "modelled" by categories in that their Betti numbers agree; our modelling can give rise to categories that are not of topological type, and rather than just involving open covers, our modelling also involve *espaces étalés*, i.e., local homeomorphisms, which accounts for the fact that there can be more than one arrow between two objects in the associated category. Section 4 discusses virtual zero extensions in more detail and their relationship to equation (1). Section 5 classifies the injectives and projectives of $\mathbb{Q}(\mathcal{C})$ for $\mathcal{C}$ finite and semi-topological, from which the functors $! \to *$ and $* \to !$ are defined. Section 6 proves that $! \to *$ is the Serre or duality functor; we note that this result is similar to duality theory for toric varieties (see [BBFK05] and the references there). Section 7 gives a necessary (but not sufficient) linear algebraic condition for $G = (! \to *)F$ to hold, based on the "local Euler characteristics" of F and G. Section 8 states the theorem that strong n-dimensionality, a compatibility condition of $! \to *$ with pulling back, is stable under arbitrary base change; fiberwise n-dimensionality is also defined and is shown to be stable under arbitrary base change; Section 9 proves that strong dimensionality is equivalent to fiberwise dimensionality. Section 10 investigates the base change morphisms, related to those used in Section 9. Appendix A formulates duality abstractly, in hopes that we might find other interesting dualities and to put the duality used in this paper on a firmer foundation.

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2 Preliminary Remarks and Notation

In this section we make some preliminary remarks regarding this paper that are either known or easy, and we fix our notation.

If $\mathcal{C}$ is a category, then $\mathrm{Ob}\,(\mathcal{C})$ denotes the objects of $\mathcal{C}$ and $\mathrm{Fl}\,(\mathcal{C})$ denotes the morphisms of $\mathcal{C}$. If $\phi \in \mathrm{Fl}\,(\mathcal{C})$, then $s\phi$ denotes the source of ϕ, and $t\phi$ denotes the target.

2.1 Adjoints to the Pullback

A finite or infinite sequence $\ldots, u_0, u_1, \ldots$ of functors is said to be a *sequence of adjoints* if we have u_i is the left adjoint of u_{i+1} for all relevant i.

If $\mathcal{C}$ is a category, then $\mathbb{Q}(\mathcal{C})$ (respectively, $\widehat{\mathcal{C}}$) denotes the category of presheaves on $\mathcal{C}$ with values in (i.e., the category of functors from $\mathcal{C}^{\mathrm{opp}}$ to) the category of finite dimensional $\mathbb{Q}$-vector spaces (respectively, the category of sets). If $u\colon \mathcal{C} \to \mathcal{C}'$ is a functor between finite categories, and $u^*\colon \mathbb{Q}(\mathcal{C}') \to \mathbb{Q}(\mathcal{C})$ is the pullback, then according to [SGA4.I.5.1], u^* has a left adjoint $u_!$ and a right adjoint u_*. We shall denote by $u^?$ (respectively, $u^!$) the left adjoint to $u_!$ (respectively, right adjoint to u_*) when they exist. [SGA4.I.5.6] shows that any of $u, u_*, u_!$ being fully faithful implies that the other two are, and that this condition is equivalent to either adjuction morphism $\mathrm{id} \to u^*u_!$ or $u^*u_* \to \mathrm{id}$ being an isomorphism.

Let us spell out u^*, u_*, the adjoint mappings, and the adjunctive morphism $\mathrm{Id} \to u_*u^*$. Let $F \in \mathbb{Q}(\mathcal{C}')$ and $G \in \mathbb{Q}(\mathcal{C})$. For $X \in \mathrm{Ob}\,(\mathcal{C})$ we have $(u^*F)(X) = F(u(X))$, and for $Y \in \mathrm{Ob}\,(\mathcal{C}')$ we have

$$(u_*G)(Y) = \varprojlim_{X;u(X)\to Y} G(X),$$

where the limit is over the category whose objects are pairs (X, m) with $m\colon u(X) \to Y$ (see [SGA4.I.5.1]). Next we describe

$$\mu\colon \mathrm{Hom}(u^*F, G) \to \mathrm{Hom}(F, u_*G);$$

if $\phi \in \mathrm{Hom}(u^*F, G)$, then we have maps

$$\phi_X\colon (u^*F)(X) = F(u(X)) \to G(X),$$

and the map

$$(\mu\phi)_Y\colon F(Y) \to \varprojlim_{X;u(X)\to Y} G(X)$$

is simply given by

$$F(Y) \to \varprojlim_{X;u(X)\to Y} F(u(X)) \quad \to \varprojlim_{X;u(X)\to Y} G(X),$$

where the first arrow is uniquely determined from the definition of limit, and the second arrow arises from applying ϕ_X to each $F(u(X))$. The quasi-inverse to μ, ν, is given on $\psi \in \operatorname{Hom}(F, u_* G)$ via

$$F(u(X)) \quad \to \quad \varprojlim_{Z;u(Z)\to u(X)} G(Z) \quad \to \quad G(X),$$

where the first arrow is given by $\psi_{u(X)}$, and the second by the canonical map of the limit onto $G(X)$ corresponding to the object (X, Id_X) (in the category over which the limit is taken). See [SGA4.I.5.1] for details of the above (there they discuss only $u_!$, where the arrows are reversed).

Setting $G = u^*F$, it follows that the adjuction morphism $\mathrm{Id} \to u_* u^*$ is given by the natural map

$$F(X) \to \varprojlim_{Z;u(Z)\to X} F(u(Z)).$$

In the above, we have implicitly touched on a number of properties of limits. Another fact we will use is that if in addition we have $v\colon \mathcal{C}' \to \mathcal{C}''$ with $\mathcal{C}''$ finite, then there is a canonical isomorphism

$$\varprojlim_{Y;v(Y)\to Z} \quad \varprojlim_{X;u(X)\to Y} G(X) \simeq \varprojlim_{X;(vu)(X)\to Z} G(X)$$

(this can be verified directly, or follows because $(vu)_*$ is canonically isomorphic to $v_* u_*$, using Yoneda's lemma and that $(vu)^* = u^* v^*$).

If $P \in \mathrm{Ob}\,(\mathcal{C})$, then $k_P\colon \Delta_0 \to \mathcal{C}$ denotes the map from the one object, one morphism category, Δ_0, to $\mathcal{C}$ sending the object of Δ_0 to P. For a $\mathbb{Q}$-vector space, V, [SGA4.I.5.1] shows that $k_{P!}V$ is isomorphic to the sheaf whose value at $Q \in \mathrm{Ob}\,(\mathcal{C})$ is

$$(k_{P!}V)(Q) = V^{\operatorname{Hom}_{\mathcal{C}}(Q,P)}; \tag{6}$$

notice that although [SGA4.I.5.1] defines $k_{P!}$ as a limit, and therefore ambiguous up to isomorphism, we shall chose $k_{P!}$ to mean equality in equation (6) (this may seem nitpicky, but it will be necessary to chose one version of $k_{P!}$

to define ! $\to *$ in Section 5); a morphism $\eta\colon Q_1 \to Q_2$ gives a map from $\mathrm{Hom}(Q_2,P)$ to $\mathrm{Hom}(Q_1,P)$, giving rise to a morphism

$$V^{\mathrm{Hom}(Q_1,P)} \to V^{\mathrm{Hom}(Q_2,P)}$$

and its transpose

$$(k_{P!}V)(\eta)\colon (k_{P!}V)(Q_2) \to (k_{P!}V)(Q_1).$$

The functor k_{P*} is the same with arrows reversed, e.g., replace $\mathrm{Hom}_{\mathcal{C}}(Q,P)$ with

$$\mathrm{Hom}_{\mathcal{C}}(P,Q) = \mathrm{Hom}_{\mathcal{C}^{\mathrm{opp}}}(Q,P)$$

(but the map $(k_{P*}V)(\eta)$ is defined directly, without the transpose).

It will be important to study how adjoint functors give rise to adjoints in the derived categories. Let a functor $u\colon \mathcal{A} \to \mathcal{B}$ have right adjoint $v\colon \mathcal{B} \to \mathcal{A}$, where $\mathcal{A},\mathcal{B}$ are Abelian categories. Let $A^\bullet$ be a complex in $\mathcal{A}$ and $B^\bullet$ one in $\mathcal{B}$. By $uA^\bullet$ we mean the complex whose i-th element is uA^i, and similarly for $vB^\bullet$. A morphism of complexes $uA^\bullet \to B^\bullet$ gives arrows $uA^i \to B^i$, that in turn give maps $A^i \to vB^i$; it is easy to check that these maps give a morphism of complexes $A^\bullet \to vB^\bullet$ that preserves homotopies. We can invert this procedure, and therefore conclude that u,v are adjoints in $\mathcal{K}(\mathcal{A}),\mathcal{K}(\mathcal{B})$ (the categories of complexes with morphisms being chain maps modulo homotopy), i.e., we have a bi-natural isomorphism in the variables $A^\bullet, B^\bullet$

$$\mathrm{Hom}_{\mathcal{K}(\mathcal{B})}(uA^\bullet, B^\bullet) \simeq \mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(A^\bullet, vB^\bullet).$$

If either $uA^\bullet$ is a complex of injectives or $B^\bullet$ is a complex of projectives, then

$$\mathrm{Hom}_{\mathcal{K}(\mathcal{B})}(uA^\bullet, B^\bullet) = \mathrm{Hom}_{\mathcal{D}}(uA^\bullet, B^\bullet),$$

where $\mathcal{D}$ is any of $\mathcal{D}(\mathcal{B}), \mathcal{D}^+(\mathcal{B}), \mathcal{D}^-(\mathcal{B}), \mathcal{D}^{\mathrm{b}}(\mathcal{B})$ as appropriate. A similar remarks holds for $\mathrm{Hom}(A^\bullet, vB^\bullet)$. We conclude (among other similar remarks) that if any element of $\mathcal{A}$ or $\mathcal{B}$ has a bounded injective resolution and a bounded projective resolution then $\underline{\underline{L}}u, \underline{\underline{R}}v$ are adjoints in $\mathcal{D}^{\mathrm{b}}(\mathcal{A}), \mathcal{D}^{\mathrm{b}}(\mathcal{B})$.

Here is another remark on adjoints that we shall use. Let $u\colon \mathcal{A} \to \mathcal{B}$ be a fully faithful functor with right adjoint, v, Then we claim that the adjunctive map $\mathrm{Id} \to vu$ is an isomorphism (as mentioned in the proof of [SGA4.I.5.6]). Indeed this follows from Yoneda's lemma and the bi-natural isomorphism in A_1, A_2

$$\mathrm{Hom}_{\mathcal{A}}(A_1,A_2) \simeq \mathrm{Hom}_{\mathcal{B}}(uA_1,uA_2) \simeq \mathrm{Hom}_{\mathcal{A}}(A_1,vuA_2).$$

More generally, let $\tilde{u}\colon \tilde{\mathcal{A}} \to \mathcal{B}$ have right adjoint, v, such that the image of v is contained in the subcategory $\mathcal{A}$ of $\tilde{\mathcal{A}}$, and with $u = \tilde{u}|_{\mathcal{A}}\colon \mathcal{A} \to \mathcal{B}$ fully faithful. We claim that the adjuctive map $\mathrm{Id} \to v\tilde{u}$ restricted to $\mathcal{A}$ is an isomorphism on each object of $\mathcal{A}$, and is the same as $\mathrm{Id} \to vu$. Indeed for $A \in \mathrm{Ob}\,(\mathcal{A})$ we have

$$\mathrm{Hom}_{\mathcal{B}}(uA, B) = \mathrm{Hom}_{\mathcal{B}}(\tilde{u}A, B) \simeq \mathrm{Hom}_{\tilde{\mathcal{A}}}(A, vB) = \mathrm{Hom}_{\mathcal{A}}(A, vB),$$

which shows that v is also a right adjoint to u with the bi-natural isomorphism $\mathrm{Hom}_{\mathcal{B}}(uA, B) \to \mathrm{Hom}_{\mathcal{A}}(A, vB)$ of the u, v adjointness being the restriction that of the $\tilde{u}, v$ adjointness. Hence the adjunctive map $A \to vuA$, which is the image of Id_{uA}, is the same as the adjunctive map $A \to \tilde{v}uA$. Finally from the above we know that $A \to vuA$ is an isomorphism.

2.2 Partial Order and Primes

Consider a semitopological category, $\mathcal{C}$ (as in the introduction, this means that any morphism from an object to itself is the identity morphism of that object). For $U, V \in \mathrm{Ob}\,(\mathcal{C})$ we write $U \le V$ or $V \ge U$ if there is a morphism from U to V. This is a semi-partial order, meaning that it is a partial order except for that we may have $U \le V$ and $V \le U$ without U and V being the same object (but then U and V must be isomorphic). If the category is *sober*, meaning that any two isomorphic objects are equal, then the semi-partial order becomes a partial order.

Throughout this paper, when we speak of "greater," "increasing chains," etc., we mean with respect to this semi-partial order.

To *factor* a morphism means to write it as a composition of two or more morphisms. A *prime* is a nonidentity morphism that cannot be factored into two nonidentity morphisms. A functor is determined by its action on the objects and prime morphisms, assuming every morphism can be factored into a finite number of primes.

2.3 Composable Morphisms

In a category, $\mathcal{C}$, we use $\mathrm{Fl}^0\,(\mathcal{C})$ to denote $\mathrm{Ob}\,(\mathcal{C})$. For any integer $i \ge 1$, we use $\mathrm{Fl}^i\,(\mathcal{C})$ to denote the set of i-tuples of morphisms $(\phi_1, \ldots, \phi_i)$ that are *composable*, meaning that $s\phi_k = t\phi_{k-1}$ for $k = 2, \ldots, i$ (so that $\phi_i \circ \cdots \circ \phi_1$ exists); in particular $\mathrm{Fl}^1\,(\mathcal{C}) = \mathrm{Fl}\,(\mathcal{C})$.

For integer $m \geq 0$, let Δ_m denote the category whose objects are $\{0, \ldots, m\}$ and with one or zero (respectively) morphisms from i to j according to whether or not $i \leq j$. We often call Δ_m the *m-dimensional simplex*. A functor, F, from Δ_m to a category $\mathcal{C}$ is determined by the m composable morphisms $F(0 \to 1), \ldots, F(m-1 \to m)$. Let $\underline{\mathrm{Fl}}^m(\mathcal{C})$ be the category whose objects are functors from Δ_m to $\mathcal{C}$ (and whose morphisms are natural transformation); clearly the objects of $\underline{\mathrm{Fl}}^m(\mathcal{C})$ can be identified with $\mathrm{Fl}^m(\mathcal{C})$.

We extend the definition of Δ_m and $\mathrm{Fl}^m(\mathcal{C})$ to $m = -1$ by setting Δ_{-1} to be the empty category, making $\underline{\mathrm{Fl}}^{-1}(\mathcal{C})$ to be a *ponctuel* category of one object and one morphism.

For integer $m \geq -1$ we define the usual $m+2$ functors $\mathrm{coface}_i\colon \Delta_m \to \Delta_{m+1}$ determined, for $m \geq 0$ and $i = 0, \ldots, m+1$, by the map on objects (since Δ_{m+1} is a partial order),

$$\mathrm{coface}_i(j) = \begin{cases} j & \text{if } j < i, \\ j+1 & \text{otherwise.} \end{cases}$$

Then $\mathrm{face}_i = \mathrm{coface}_i^*$ gives rise to the usual simplicial complex

$$\mathrm{Fl}^{-1}(\mathcal{C}) \leftarrow \mathrm{Fl}^0(\mathcal{C}) \leftleftarrows \mathrm{Fl}^1(\mathcal{C}) \Lleftarrow \mathrm{Fl}^2(\mathcal{C}) \cdots,$$

where the faces of $(\phi_1, \ldots, \phi_m) \in \mathrm{Fl}^m(\mathcal{C})$ (in order, from face_0 to face_m,) are

$$(\phi_2, \ldots, \phi_m),\ (\phi_2 \circ \phi_1, \phi_3, \ldots, \phi_m),\ (\phi_1, \phi_3 \circ \phi_2, \phi_4, \ldots),\ \ldots,$$

$$(\phi_1, \ldots, \phi_{m-2}, \phi_m \circ \phi_{m-1}),\ (\phi_1, \ldots, \phi_{m-1})$$

(see, for example, [SGA4.V.2.3.6]).

2.4 Simiplicial Complex, Simplicial Hom, and Graphs

In the previous subsection we have described a map taking a category and returning a simplicial complex; denote this map u. Furthermore, there is a map, v, that associates to each simplicial complex its associated graph, by forgetting about the sets of dimension two and greater. The map $v \circ u$ is the usual forgetful functor from categories to graphs; its left adjoint is the free category associated to a graph, which associates to a graph the category whose morphisms are walks in the graph (and objects being the vertices of the graph).

For example, the free category associated to a directed path of length n is the category Δ_n. A category is isomorphic to a free category iff it is a *unique factorization domain*, i.e., each nonidentity morphism can be factored as a finite number of primes in exactly one way.

Given a category, $\mathcal{C}$, and an additive category, $\mathcal{D}$, we define a category $\mathrm{SHom}\,(\mathcal{C},\mathcal{D})$, the *simplicial Hom of* $\mathcal{C}$ *to* $\mathcal{D}$ as follows: its objects are the set theoretical maps from $\mathrm{Ob}\,(\mathcal{C})$ to $\mathrm{Ob}\,(\mathcal{D})$; given objects F, G, an element of $\mathrm{Hom}(F,G)$ is a map $\alpha\colon \mathrm{Fl}\,(\mathcal{C}) \to \mathrm{Fl}\,(\mathcal{D})$ such that $s \circ \alpha = F \circ s$ and same with t replacing s and G replacing F; in other words, for each morphism $\phi \in \mathrm{Fl}\,(\mathcal{C})$, $\alpha\phi$ is an arrow $Fs\phi \to Gt\phi$. The composition is defined via

$$(\beta\alpha)(\phi) = \sum_{\phi_2\phi_1=\phi} (\beta\phi_2)(\alpha\phi_1).$$

The morphisms and their compositions can be viewed as a generalization of matrices with matrix multiplication over $\mathcal{D}$ indexed in the objects of $\mathcal{C}$. For an object, F, we have Id_F is the map $\mathrm{Id}_F(\mathrm{Id}_X) = \mathrm{Id}_{F(X)}$ and $\mathrm{Id}_F(\phi) = 0$ if $s\phi \neq t\phi$.

2.5 The Grossière Topology

In this subsection we show that the category of sheaves of sets of any finite semitopological Grothendieck topology, $(\mathcal{C}, J)$, is equivalent to the category of presheaves of sets (on a certain full subcategory of $\mathcal{C}$).

Let $E = (\mathcal{C}, J)$ be a Grothendieck topology or site. We say that $X \in \mathrm{Ob}\,(\mathcal{C})$ is *gross* if $J(X) = \{X\}$ (i.e., the only element of $J(X)$ is the sieve that is the entirety of $\mathcal{C}/X$). Recall that the *grossière* (meaning gross or coarse) topology ([SGA4.II.1.1.4]) for a category $\mathcal{C}$ is the Grothendieck topology for which each object is gross; in this topology a sheaf is the same thing as a presheaf (as defined here and by Grothendieck, [SGA4.I.1.2]).

If U is an open set in a topological space, then U is gross iff it is irreducible, i.e., iff U is not the union of its proper open subsets, iff there is a point, $p \in U$, such that any open set containing p contains U.

Theorem 2.1 *Let $E = (\mathcal{C}', J')$ be a finite Grothendieck topology. The gross objects of $\mathcal{C}'$ determine a full subcategory, $\mathcal{C}$; let $u\colon \mathcal{C} \to \mathcal{C}'$ be the inclusion. Then u^* gives an equivalence of categories between $\widehat{\mathcal{C}}$ and the category of sheaves on E.*

Proof Let J be the topology induced by u on $\mathcal{C}$. First we show that J is the *grossière* topology. Let $R \in J(Y)$. Since u is fully faithful we have $u^*u_!R = R$; it follows that $(u_!R)(Y) = R(Y)$. But the image of $u_!R \to Y$ is *bicouvrant* by [SGA4.III.3.2], and in particular $(u_!R)(Y) = \{\mathrm{Id}_Y\}$. It follows that $R(Y) = \{\mathrm{Id}_Y\}$, and so $R = Y$. Hence $J(Y) = \{Y\}$, and J is the *grossière* topology.

We finish by showing that u satisfies condition (i) of the hypothesis of the Comparison Lemma ([SGA4.III.4.1]); condition (ii) then follows, which is the claim of our theorem. If $\{H_i \to K\}$ is a *couvrante* family, and $L_i \to H_i$ is *couvrant* for each i, then $\{L_i \to K\}$ is a *couvrante* family, by [SGA4.II.5.1.ii]. This remark allows us to take any *couvrante* family, $\{Y_i \to X\}$, where the Y_i are objects, take any Y_k that is not gross and maximal among the non-gross, and replace it with objects less than it (here "maximal" and "less" refer to the partial order $V \leq W$ if there is a morphism from V to W). It follows (since $\mathcal{C}'$ is finite) that any object in $\mathcal{C}'$ can be covered[5] by morphisms from gross objects.

□

2.6 Topological Notions

The following notions agree with [SGA4.IV] in the case where a category is endowed with the *grossière* topology, which is our running assumption for most of this paper. A point in a sober, finite, semitopological category, $\mathcal{C}$, is an object of $\mathcal{C}$ (see [SGA4.IV.6.8.6]). An *open set* of $\mathcal{C}$ (see [SGA.IV.8.4.4]) is a *sieve*, i.e., a full subcategory, $\mathcal{C}'$, of $\mathcal{C}$, such that if $f \in \mathrm{Fl}(\mathcal{C})$ and $tf \in \mathrm{Ob}(\mathcal{C}')$, then $sf \in \mathrm{Ob}(\mathcal{C}')$; a sieve is determined by its objects, and we sometimes identify the sieve with its set of objects (if no confusion will arise). *Closed sets* and *cosieves* are defined dually, i.e., as open sets and sieves (respectively) in $\mathcal{C}^{\mathrm{opp}}$. The *complement* of a subcategory, $\mathcal{C}$, of $\mathcal{C}'$ is the full subcategory of $\mathcal{C}'$ whose objects are $\mathrm{Ob}(\mathcal{C}') \setminus \mathrm{Ob}(\mathcal{C})$.

[5] The word *recouvrement*, used in [SGA4.II.5.1], does not appear to be defined up to that point; however, it is clear from the proof (especially where $pn = qn$ implies that the kernel of p, q is a *couvrant* sieve) (and from [SGA4.V.2.4.3]) that a set of objects, $\mathcal{S}$, is a *recouvrement* of X if there is a family of morphisms with sources in $\mathcal{S}$ and target X that is a *couvrante* family for X. (Also, there is no word *couvrement* in French, so in making a noun out of *couvrir* one must choose between *recouvrement* and *couverture*, the latter not sounding very mathematical.)

If $j\colon U \to X$ is the inclusion of an open subcategory, U, of a category, X, and $G \in \mathbb{Q}(U)$, then $j_!G$ is the usual extension by 0 of G; if $F \in \mathbb{Q}(X)$, then the left adjoint, $j^?$, of $j_!$ satisfies

$$(j^? F)(V) \simeq \{\ell \in F(V)^* \mid \ell(F\phi) = 0 \quad \forall \phi\colon V \to A,\ \text{with } A \notin U\}^*$$

$$\simeq F(V) \Big/ \bigoplus_{\phi\colon V\to A,\ \text{with } A\notin U} \operatorname{im}(F\phi) \quad \simeq \operatorname{coker} \bigoplus_{\phi\colon V\to A,\ \text{with } A\notin U} F\phi$$

Similarly if $i\colon Z \to X$ is a closed inclusion, then i_* is the usual extension by 0, and has right adjoint, $i^!$ with

$$(i^! F)(V) \quad \simeq \quad \ker \bigoplus_{\phi\colon A\to V,\ \text{with } A\notin Z} F\phi$$

(often called "sections supported on Z").

2.7 Simple Duality

For a presheaf, F, of finite dimensional $\mathbb{Q}$ vector spaces on a category, $\mathcal{C}$, we define the presheaf of $\mathcal{C}^{\text{opp}}$, F^{dl} as follows: first,

$$F^{\text{dl}}(U) = \operatorname{Hom}_{\mathbb{Q}}(F(U), \mathbb{Q});$$

second, a morphism, $\phi\colon U \to V$ in $\mathcal{C}$ corresponds to a morphism $\phi^{\text{opp}}\colon V \to U$ in $\mathcal{C}^{\text{opp}}$, and we define $F^{\text{dl}}\phi^{\text{opp}}$ to be the map dual to $F\phi$.

Theorem 2.2 *The functor "dual" is exact, involutive, and exchanges injectives for projectives and vice versa. Furthermore, for a full inclusion of categories, k, we have $(k_*) \circ \text{dl} = \text{dl} \circ k_!$.*

By passing to the "dual" of a sheaf on the opposite category, we can often prove two theorems at once.

2.8 Vector Bundles

By a *vector bundle* we mean an $F \in \mathbb{Q}(\mathcal{C})$ such that $F\phi$ is an isomorphism for all $\phi \in \operatorname{Fl}(\mathcal{C})$. Associated to F is its *dual bundle*, $F^\vee$, such that for $X \in \operatorname{Ob}(\mathcal{C})$ we have $F^\vee(X)$ is the dual space to $F(X)$, and for $\phi \in \operatorname{Fl}(\mathcal{C})$ we have $F^\vee\phi$ is the inverse of the dual to $F\phi$. A *line bundle* or *invertible sheaf* is a line bundle for which $F(X) \simeq \mathbb{Q}$ for each $X \in \mathrm{Ob}\,(\mathcal{C})$.

This notion of vector bundle is justified as follows. If $X \in \mathrm{Ob}\,(\mathcal{C})$, then by *localization at* X we mean the "source" map $j_X \colon \mathcal{C}/X \to \mathcal{C}$ ([SGA4.I.5.10–12,III.5,IV.8]). (So if $\mathcal{C}$ is topological, then $\mathcal{C}/X$ can be identified with the open subset associated with X, and j_X is the usual inclusion of categories.) By a *vector bundle* we mean an $F \in \mathbb{Q}(\mathcal{C})$ that is locally trivial, i.e., such that for each $X \in \mathrm{Ob}\,(\mathcal{C})$ we have $j_X^* F$ is isomorphic to a number of copies of $\mathbb{Q}$; it is easy to see that this is equivalent to the definition of vector bundle in the previous paragraph.

If F is a vector bundle, then there is a *dual vector bundle*, $F^\vee$, given by

$$X \mapsto F^\vee(X) = \mathrm{Hom}(j_X^* F, \mathbb{Q}),$$

and for $\phi \colon X \to Y$ in $\mathcal{C}$ we determine $F^\vee\phi$ by the functor $\mathcal{C}/Y \to \mathcal{C}/X$ arising from ϕ. This definition is equivalent to the previous one.

We remark that for any vector bundle, F, in $\mathcal{C}$, and $G, H \in \mathbb{Q}(\mathcal{C})$, we have

$$\mathrm{Ext}^i(F \otimes G, H) \simeq \mathrm{Ext}^i(G, H \otimes F^\vee),$$

since both left- and right-hand-sides are delta functors in H that are isomorphic for $i = 0$. (More generally, if $A^\bullet, B^\bullet$ are bounded complexes, then $\mathrm{Hom}(F \otimes A^\bullet, B^\bullet) \simeq \mathrm{Hom}(A^\bullet, B^\bullet \otimes F^\vee)$ in the category, $\mathcal{K}^{\mathrm{b}}(\mathbb{Q}(\mathcal{C}))$, of bounded complexes over $\mathbb{Q}(\mathcal{C})$, whose morphisms are chain maps modulo homotopy.)

2.9 Abstract Principles

2.9.1 Usage of the Axiom of Choice

Some functors constructed in this paper, notably some quasi-inverses and Serre functors, have freedom in their definition and require choices to be made definite. At first glance it seems we require the Axiom of Choice (e.g., Axiom (UB) of [SGA4.I.1], page 3) for this. However, in practice we are interested in the behavior of these functors only on a finite number of objects (and morphisms between these objects). It is not hard to see that it suffices to apply the Axiom of Choice to subcategories with a finite number of objects, whereby the axiom of chioce is not needed. Let us give an example.

Say that $F \colon \mathcal{X} \to \mathcal{Y}$ is fully faithful and essentially surjective, and say that we wish to construct a quasi-inverse, G, to F, without invoking the

Axiom of Choice. Say that we are interested to know G on only a subcategory of $\mathcal{Y}$, $\mathcal{Y}'$, that has only finitely many objects. Then it is easy to see that we can find subcategories $\mathcal{X}'', \mathcal{Y}''$ (respectively) of $\mathcal{X}, \mathcal{Y}$ (respectively) such that (1) each has finitely many objects, (2) $\mathcal{Y}'$ is a subcategory of $\mathcal{Y}''$, (3) F maps $\mathcal{X}''$ to $\mathcal{Y}''$ fully faithfully and essentially surjectively. The quasi-inverse of F restricted to $\mathcal{X}'', \mathcal{Y}''$ can be constructed with only a finite number of choices.

In the last paragraph, one can say that F (as a functor $\mathcal{X} \to \mathcal{Y}$) is an *extension* of $F|_{\mathcal{X}''}$ (F's restriction as a functor $\mathcal{X}'' \to \mathcal{Y}''$), or that $F|_{\mathcal{X}''}$ is a *restriction* of F. Functors are partially ordered with respect to extension. The last paragraph makes it look like we need to fix $\mathcal{Y}'$ or $\mathcal{Y}''$ once and for all. However, if one sees that one needs a quasi-inverse to F on a larger subcategory than $\mathcal{Y}'$ or $\mathcal{Y}''$, then one is free to extend the quasi-inverse a finite number of times (to successively larger categories provided each has a finite number of objects) using only finite choices. We won't state a formal result, just note that we need to extend categories so that we can find choice data (see Section 2.9.3 below) for the new objects of $\mathcal{Y}$ on which we want the quasi-inverse defined.

2.9.2 Equivalence of Compositions of Functors

If $E\colon \mathcal{C} \to \mathcal{C}$ is an equivalence of categories, with ϕ the invertible natural transformation from $\mathrm{Id}_{\mathcal{C}}$ to E, and if F, G are functors such that FEG exists, then $FG \simeq FEG$ by horizontally composing $\mathrm{Id}_F \phi \mathrm{Id}_G$ on $F\mathrm{Id}_{\mathcal{C}}G$. It follows (by vertical composition) that one composition of functors is isomorphic to another iff the same is true when we insert arbitrary equivalences of categories into the compositions.

2.9.3 Representative Subcategories

We say that a full subcategory, $\mathcal{C}'$, of a category, $\mathcal{C}$, is a *representative subcategory* if every object of $\mathcal{C}$ is isomorphic to at least one object of $\mathcal{C}'$; by *choice data,* (Z, ι) for such a situation we mean a map $Z\colon \mathrm{Ob}\,(\mathcal{C}) \to \mathrm{Ob}\,(\mathcal{C}')$ and $\iota\colon \mathrm{Ob}\,(\mathcal{C}) \to \mathrm{Fl}\,(\mathcal{C})$ such that for each $X \in \mathrm{Ob}\,(\mathcal{C})$, $\iota(X)$ is an isomorphism from X to $Z(X)$.

Given a representative subcategory, choice data always exists provided we are willing to invoke the Axiom of Choice (but see Section 2.9.1). Alternatively, sometimes the data, or at least part of it, can be made explicit. Finally, we remark that sometimes we want the choice data, especially the morphisms $\iota(X)$, to satisfy additional constraints (for example, in our construction of $! \to *$).

In a number of situations in this paper, notably with the derived category, it is much simpler to work with a representative subcategory (in defining functors and natural transformations). General principles say that we can extend the functors and natural transformations to the original category. Here we carefully state these general principles, at least those that we use in this paper.

Functor extensions Given a representative subcategory, $\mathcal{C}'$, of $\mathcal{C}$, with choice data (Z,ι), and given a functor $F\colon \mathcal{C}' \to \mathcal{E}$, we define $F'\colon \mathcal{C} \to \mathcal{E}$, called *F's extension to* $\mathcal{C}$ via:

$$F'(X) = F\bigl(Z(X)\bigr)$$

for $X \in \mathrm{Ob}\,(\mathcal{C})$, and for $\phi\colon X_1 \to X_2$ in $\mathrm{Fl}\,(\mathcal{C})$ set

$$F'(\phi) = F\bigl(\iota(X_2) \circ \phi \circ \iota(X_1)^{-1}\bigr).$$

This construction is absolutely standard (it is essentially how quasi-inverses are constructed). It is standard and easy that if F'' is an extension formed by other choice data, then $F' \simeq F''$.

Natural Transformation Extensions Consider a representative subcategory, $\mathcal{C}'$, of $\mathcal{C}$, with choice data (Z,ι). Let F,G be two functors from $\mathcal{C}$ to $\mathcal{E}$. Let ϕ be a natural transformation from $F|_{\mathcal{C}'}$ (i.e., F restricted to $\mathcal{C}'$) to $G|_{\mathcal{C}'}$. We can extend ϕ to a morphism $\phi'\colon F \to G$ by setting

$$\phi'(X) = \bigl(G\iota(X)\bigr)^{-1}\phi\bigl(Z(X)\bigr)F\iota(X).$$

We easily verify that ϕ' is a natural transformation, since for $f\colon X \to Y$ in $\mathcal{C}$, each of the small squares in the diagram below commute:

$$\begin{array}{ccc}
FX & \xrightarrow{\phi'(X)} & GX \\
{\scriptstyle F\iota X}\downarrow & & \downarrow{\scriptstyle G\iota X} \\
F(Z(X)) & \xrightarrow{\phi(Z(X))} & G(Z(X)) \\
{\scriptstyle F(\iota(Y)f\iota^{-1}(X))}\downarrow & & \downarrow{\scriptstyle G(\iota(Y)f\iota^{-1}(X))} \\
F(Z(Y)) & \xrightarrow{\phi(Z(Y))} & G(Z(Y)) \\
{\scriptstyle F\iota^{-1}Y}\downarrow & & \downarrow{\scriptstyle G\iota^{-1}Y} \\
FY & \xrightarrow{\phi'(Y)} & GY
\end{array}$$

2.10 The Standard Resolution

Let $\mathcal{C}$ be a category. Let $\mathcal{F} \subset \mathrm{Fl}\,(\mathcal{C})$ be such that $\phi \notin \mathcal{F}$ implies that $\phi = \mathrm{Id}_X$ for some object X such that the only morphism from X to itself is the identity. For example, if $\mathcal{C}$ is semitoplogical, then $\mathcal{F}$ can be any collection of morphisms that includes all nonidentity morphisms. Also, regardless of $\mathcal{C}$, we can always take $\mathcal{F} = \mathrm{Fl}\,(\mathcal{C})$, which is the usual standard resolution ([SGA4.V.2.3.6]).

Let $\mathcal{F}^i$ be the composable i-tuples of morphisms. For $F \in \mathbb{Q}(\mathcal{C})$ set

$$P_i = P_i(F) = \bigoplus_{\phi \in \mathcal{F}^i} k_{s(\phi)!} F\bigl(t(\phi)\bigr)$$

and

$$I^i = I^i(F) = \bigoplus_{\phi \in \mathcal{F}^i} k_{t(\phi)*} F\bigl(s(\phi)\bigr).$$

The structure of the $\mathcal{F}^i$ as a simplicial set (see Subsection 2.3 or [SGA4.V.2.3.6], for example) gives complexes[6]

$$\cdots \to P_1 \to P_0 \to F$$

[6]Note that morally speaking we are saying that P_{-1} is F. We are not entirely sure why. Perhaps, since P_i is a sum of $k_{s!}k_t^* F$, when there is no s, t the k's should be omitted, leaving just F? We leave this to experts on the empty category, Δ_{-1}...

and

$$F \to I^0 \to I^1 \to \cdots$$

We claim these complexes give a projective resolution and an injective resolution respectively. Too see this, by simple duality it suffices to check the case of the P_i's. To check the exactness at $Y \in \mathrm{Ob}\,(\mathcal{C})$, it suffices to find a chain homotopy K such that $dK + Kd = \mathrm{id}$. Now

$$P_i(Y) = \bigoplus_{\phi \in \mathcal{F}^i} F\bigl(t(\phi)\bigr)^{\mathrm{Hom}\left(Y, s(\phi)\right)} = \bigoplus_{\mu;\phi_1,\ldots,\phi_i} F\bigl(t(\phi_i)\bigr),$$

where the rightmost direct sum ranges over $\mu \in \mathrm{Fl}\,(\mathcal{C})$ and $\phi_1, \ldots, \phi_i \in \mathcal{F}$ such that $\mu, \phi_1, \ldots, \phi_i$ is composable. If $w \in P_i(Y)$ we define Kw via its components as

$$(Kw)_{\alpha;\mu,\phi_1,\ldots,\phi_i} = \begin{cases} w_{\mu;\phi_1,\ldots,\phi_i} & \text{if } \alpha = \mathrm{id}_Y, \\ 0 & \text{otherwise.} \end{cases}$$

If $w \in F(Y)$ we define $(Kw)_{\mathrm{id}_Y;} = w$ and define Kw to vanish on all other components. We easily verify $dK + Kd = \mathrm{id}$ on the complex in question.

In particular, let $\mathcal{C}$ be a semitopological category, and let the *dimension* of $\mathcal{C}$, denoted $d = d(\mathcal{C})$, be the length of the longest sequence of composable nonidentity morphisms in $\mathcal{C}$. Alternatively, $d + 1$ is the length of the longest chain in the partially ordered set of objects. Alternatively d is the *topological dimension*, meaning that $d + 1$ is the length of the longest proper chain of closed (or open) irreducible sets. If $\mathcal{C}$ is finite, then d is finite, and at most the number of objects minus one. If $\mathcal{F}$ as above is the set of all nonidentity morphisms, then $\mathrm{Fl}^i\,(\mathcal{C})$ is empty for $i > d$; hence $\mathcal{C}$ has homological dimension at most d (see [GM03], meaning for any sheaves F, G we have $\mathrm{Ext}^i(F, G) = 0$ for $i > d$).

2.11 Other Resolutions

For an arbitrary finite, semitopological category, $\mathcal{C}$, there are *greedy resolutions* of $F \in \mathbb{Q}(\mathcal{C})$, constructed as follows. For each $X \in \mathrm{Ob}\,(\mathcal{C})$ we set

$$G_X = \ker \bigoplus_{\phi\colon Y \to X,\ Y \neq X} F\phi,$$

in other words those elements of $F(X)$ that get sent to zero by each $F\phi$ with ϕ having target X and source not equal to X. (If X is an initial element, then $G_X = F(X)$.) We choose an element in

$$\iota_X \in \mathrm{Hom}(F(X), G_X)$$

that restricts to the identity on G_X; we therefore get for each X an element of $\mathrm{Hom}(F, k_{X*}G_X)$. This gives rise to a map

$$\iota\colon F \to I = \sum_{X \in \mathrm{Ob}(\mathcal{C})} k_{X*}G(X)$$

that we claim is initial in the category of inclusions of F into an injective; indeed, let $\iota'\colon F \to I'$ where I' is a sum $k_{X*}V_X$; note that we have a canonical isomorphism

$$V_X \simeq \ker \bigoplus_{\phi\colon Y \to X,\ Y \neq X} I'\phi.$$

Then ι' gives an injection $G_X \to V_X$, giving a morphism $k_{X*}G_X \to k_{X*}V_X$, and a morphism $\nu\colon I \to I'$. We easily show that $\nu \circ \iota = \iota'$ and ν is uniquely determined by this constraint, by structural induction on $\mathcal{C}$ (i.e., we show this at initial objects, and then remove the initial objects from $\mathcal{C}$, passing to the "kernel" of F by $F\phi$ with ϕ having source in an initial object (i.e., $i^! F$ with i the closed inclusion), and use induction).

By a *greedy* injective resolution of F we mean any inductive resolution

$$F \to I^0 \to I^1 \to \cdots$$

with I^j obtained greedily from the cokernel of the map to I^{j-1}. Greedy resolutions are often much more efficient in practice (for example, computing the cohomology of the examples in Section 3) than the standard resolution.

For reasons we do not understand, it is often (but not always) the case that greedy resolutions of $\mathbb{Q}$ and other sheaves (especially in "geometric" examples) have the property that a summand $k_{X*}V_X$ for a fixed X appears in only one of the I^j (in some sort of "rank" order).

Greedy projective resolutions can be defined similarly.

Next we describe a special resolution for finite, semitopological categories isomorphic to a free category. If $\mathcal{C}$ is a free category, formed from the directed graph, G, then the primes of $\mathcal{C}$ are just the morphism corresponding to G edges. We claim that any $F \in \mathbb{Q}(\mathcal{C})$ has a projective resolution

$$0 \to P_1 \to P_0 \to F \to 0,$$

where

$$P_1 = \bigoplus_{\phi \in E(G)} k_{s(\phi)!} F\bigl(t(\phi)\bigr), \qquad P_0 = \bigoplus_{X \in V(G)} k_{X!} F(X),$$

where $V(G), E(G)$ are, respectively, the vertices and edges of G. Indeed, it suffices to find for each $Y \in \mathrm{Ob}\,(\mathcal{C})$ a chain homotopy, K, with $dK + Kd = \mathrm{Id}$ on the above sequence localized at Y. Note that

$$P_0(Y) \simeq \bigoplus_{X \in V(G)} F(X)^{\mathrm{Hom}(Y,X)},$$

so that a $P_0(Y)$ element, w, is equivalent to giving for each $\nu \in \mathrm{Hom}(Y, X)$ (for any X) a "component," $w^\nu \in F(X)$. So let $K_{-1}\colon F(Y) \to P_0(Y)$ be defined by mapping $v \in F(Y)$ to v in the component $F(Y)^{\mathrm{Id}_Y}$ and zero elsewhere. Clearly $dK + Kd = \mathrm{Id}$ on $F(Y)$. Next note that

$$P_1(Y) \simeq \bigoplus_{\phi \in E(G)} F\bigl(t(\phi)\bigr)^{\mathrm{Hom}(Y,s(\phi))},$$

so a $P_1(Y)$ element, v, is equivalent to specifying for each prime ϕ (i.e., edge of G) and each $\mu \in \mathrm{Hom}(Y, s(\phi))$ a "component," $v^{\mu,\phi} \in F(t\phi)$. Set $K_0\colon P_0(Y) \to P_1(Y)$ via

$$(Kw)^{\mu,\phi} = \sum_{\alpha \in \mathrm{Fl}(\mathcal{C})} (F\alpha) w^{\phi \circ \mu \circ \alpha}.$$

Using unique factorization we easily verify that $dK + Kd = \mathrm{Id}$ on $P_0(Y)$ and $P_1(Y)$. For example, $Kd\colon P_1(Y) \to P_1(Y)$ is given by

$$(Kdw)^{\mu,\phi} = \sum_{\mu'\phi'\alpha = \mu\phi} (F\alpha) w^{\mu',\phi'} - \sum_{\mu'\phi'\alpha = \mu} (F\alpha\phi) w^{\mu',\phi'},$$

where $\alpha \in \mathrm{Fl}\,(\mathcal{C})$ and ϕ, ϕ' as before (and notation $w^{\mu',\phi'}$ as before); the equation $\mu'\phi'\alpha = \mu\phi$ can be solved by either $\alpha = \mathrm{Id}$ or $\alpha = \beta\phi$ for some β; the $\alpha = \mathrm{Id}$ solution gives $(Kdw)^{\mu,\phi}$ the summand $w^{\mu,\phi}$, where each solution where $\alpha = \beta\phi$ gives rise to a solution $\mu'\phi'\beta = \mu$, causing a cancellation.

2.12 Left to Right, Right to Left, and Serre Functors

Here we summarize the discussion of Appendix A, for the special case used in this paper.

Let $\mathcal{D}_1$ be a category whose Hom sets can be given the structure of a finite dimensional $\mathbb{Q}$-vector space (see Appendix A for what this entails; in this paper $\mathcal{D}_1$ will be of the form $\mathcal{D}^{\mathrm{b}}(\mathbb{Q}(\mathcal{C}))$, the derived category of bounded $\mathbb{Q}(\mathcal{C})$ complexes, with $\mathcal{C}$ finite and semitopological). For $B \in \mathrm{Ob}\,(\mathcal{D}_1)$, we denote by $(\mathrm{L} \to \mathrm{R})B$ or $B^{\mathrm{L}\to\mathrm{R}}$, called B *left-to-right*, the functor:

$$A \mapsto \mathrm{Hom}(B, A)^*.$$

If this functor is representable, we denote (by minor abuse of notation) by $(\mathrm{L} \to \mathrm{R})B$ or $B^{\mathrm{L}\to\mathrm{R}}$ any object representing. If $B^{\mathrm{L}\to\mathrm{R}}$ is representable for any B, then Yoneda's lemma implies that $(\mathrm{L} \to \mathrm{R})$ (called left-to-right) extends to a (covariant) functor on $\mathcal{D}_1$ (see Appendix A). The left-to-right functor, if exists, has also been called the *Serre functor* (see [BK89, BO01, BLL04], for example) in the context of the derived category. (We prefer left-to-right and later $! \to *$ as names, since they are more suggestive to our interests.)

The right-to-left functor is defined analogously, and is a pseudoinverse of the left-to-right functor (when they are representable). In this paper we shall give a simple construction of the Serre or left-to-right functor for $\mathcal{D}^{\mathrm{b}}(\mathbb{Q}(\mathcal{C}))$ as above. We also remark that if F, G are adjoints in $\mathcal{D}_1, \mathcal{D}_2$, i.e.,

$$\mathrm{Hom}_{\mathcal{D}_2}(FA, B) \simeq \mathrm{Hom}_{\mathcal{D}_1}(A, GB)$$

(an isomorphism natural in A, B), then it is easy to see that G has a left adjoint

$$(\mathrm{L} \to \mathrm{R})_{\mathcal{D}_2} F (\mathrm{R} \to \mathrm{L})_{\mathcal{D}_1}, \tag{7}$$

provided the appropriate left-to-right and right-to-left functors exist (see Appendix A or [BLL04], Remark 1.13). So when left-to-right and right-to-left functors exist for both categories, and adjoint pair F, G can be extended indefinitely on the left and right to a sequence of adjoints.

2.13 Topological Spaces with a Group Action

We claim that the topos of a Grothendieck topology on a finite, semitopological category, $\mathcal{C}$, can be described as the category of G-invariant sheaves of sets on a finite topological space, X, with a G action, for some (finite) group, G. Indeed, we may assume $\mathcal{C}$ has its *grossière* topology. The graph underlying $\mathcal{C}$ has a covering space (see [Fri93]) in which all multiple edges are separated in the cover; by Galois theory of graphs (see [Fri93], for example) the covering graph has a Galois covering[7]. This Galois cover inherits a composition law from $\mathcal{C}$, and therefore comes with the structure of a category, $\mathcal{C}'$. If G is the Galois group of the graph of $\mathcal{C}'$ over that of $\mathcal{C}$, then a sheaf of sets on $\mathcal{C}$ is the same thing as a G-invariant one on $\mathcal{C}'$. But the underlying graph of $\mathcal{C}'$ covers a graph with at most one edge between any two vertices, so $\mathcal{C}'$ is of topological type and yields a topological space.

So, in a sense, we can always replace a finite, semitopological category by a topological one with a finite group action.

3 Examples of Categories

A topological model involves a category, and we wish to give ways of finding interesting examples of categories.

First we describe how interesting categories arise from geometry. Say that a finite open covering $\{U_i\}$ of a topological space or manifold, M, is a *good cover* (respectively, *pretty good cover*) if (1) all U_i have the cohomology of a point (i.e., the same Betti numbers), and (2) each intersection $U_i \cap U_j$ equals some U_k (respectively, is the union of some of the U_k's). (Our definition of good cover is related to the nerve of a good cover in the sense of [BT82]; our pretty good covers can be used to form hypercovers as in [SGA4.V.7.3–4].) It is known that good and pretty good covers can be used to compute the Betti numbers of a space (see [SGA4.V.7.3–4], [BT82, Seg68, DI04], etc.). In the following subsection we will study generalized coverings, via *espaces étalés*, giving categories that are not of topological type.

Second, in the subsection thereafter, we give specific categories arising from manifolds and some just arising from more combinatorial considerations.

Last we remark that there are general constructions to create new categories out of old ones, such as the fiber product.

3.1 Espaces Étalés

Here we gather some facts on *espaces étalés*.

[7]The proof of this theorem in [Fri93] is incorrect; the following argument provides a correct proof. If $H_1 \to H_2$ is a covering map of graphs of degree n, then the n-fold fiber product $H_1 \times_{H_2} H_1 \times_{H_2} \cdots$ has its largest connected component of degree $n!$ over H_2, and this component is a Galois cover. See [Gro77].

A morphisms $j\colon X\to Y$ of topological spaces is an *espace étalé (over Y)* iff it is a *local homeomorphism*, meaning that for every $p\in X$ there is an open U containing p such that $j(U)$ is open and $j|_U\colon U\to j(U)$ is an isomorphism. If $j_i\colon X_i\to Y$ for $i=1,2$ are two *espaces étalés*, then a morphism from j_1 to j_2 is a morphism $\phi\colon X_1\to X_2$ such that $j_2\phi=j_1$; such a ϕ is necessarily an *espace étalé*. Also the fiber product of j_1 and j_2 exists and is an *espace étalé* $X_1\times_Y X_2\to Y$.

(The category of *espaces étalés* over Y is equivalent to the category of sheaves of sets over Y.)

If $U\subset X$ is an open subset of a topological space, X, we write j_U for the inclusion $U\to X$; j_U is an *espace étalé*.

If $j\colon X\to Y$ is an *espace étalé*, and $U\subset X$ is open, then $j(U)$ is open. The pullback, j^*, acting on sheaves of $\mathbb{Q}$-vector spaces over Y to those on X, has an exact left adjoint, $j_!$, given as the sheaf associated to the presheaf

$$(j_!^{\mathrm{pre}}F)(U)=\bigoplus_{\phi\colon j_U\to j}F(\phi(U)) \tag{8}$$

(see [SGA4.IV.11.3.1][8]). It easily follows that the stalk $(j_!F)_p$ (or, equivalently, $(j_!^{\mathrm{pre}}F)_p$) is isomorphic to

$$\bigoplus_{q\text{ s.t. }j(q)=p}F_q.$$

Since $j_!$ has an exact right adjoint (namely j^*), $j_!$ takes projectives to projectives. It follows that

$$\mathrm{Ext}^i(j_!F,G)=\mathrm{Ext}^i(F,j^*G).$$

3.2 Finite Categories Arising from Topological Spaces

In this subsection we wish to describe how finite categories can arise from topological spaces in a natural way such that the cohomology of the category

[8]Proving that the sheaf associated to the presheaf in equation (8) really is the left adjoint to j^* makes a nice exercise. Indeed, an element of $\mathrm{Hom}(j_!F,G)$ gives a Hom from the direct sum of $F(\phi(U))$'s to $G(U)$, or equivalently a product of the individual Homs; if $V\subset X$ is open with $j|_V\colon V\to j(V)$ an isomorphism, then $j^*G(V)=V\times_Y G=j(V)\times_Y G=G(j(V))$ and we have a $\phi\colon j_{j(V)}\to j$ with $\phi(j(V))=V$, giving an element of $\mathrm{Hom}(F(V),G(j(V)))=\mathrm{Hom}(F(V),j^*G(V))$. We need to check that these $\mathrm{Hom}(F(V),j^*G(V))$ agree on overlaps, and that the resulting map from Hom sets is a functorial bijection.

agrees with that of the space. We shall give a theory that, among other things, gives categories that are not of topological type. Let us start with an example.

Consider the open real intervals $U_1 = (-0.1, 0.1)$ and $U_2 = (-0.1, 1.1)$. If $S^1 = \mathbb{R}/\mathbb{Z}$, then the natural map $\mathbb{R} \to S^1$ induces natural maps $\iota_i \colon U_i \to S^1$. There are two maps from ι_1 to ι_2 (namely addition by either 0 or 1). A sheaf on S^1 pulls back to one on U_2, and a sheaf on U_2 arises as a pullback precisely when its two pullbacks to U_1 agree. We claim that this fact and the fact that U_1 and U_2 are both contractible implies that the cohomology of S^1 agrees with that of $\mathcal{U}$, where $\mathcal{U}$ is the category whose objects are $\{\iota_1, \iota_2\}$ and with two morphisms from ι_1 to ι_2 (and those are the only nonidentity morphisms). We shall give a general principle to this effect.

Let

$$\cdots \,\substack{\to\\\to\\\to}\, M_1 \rightrightarrows M_0 \to M_{-1} = M$$

be a simplicial topological space, with all arrows being *espaces étalés*. Let $j_i \colon M_i \to M$ be the composite arrow. We claim that the two conditions:

$$\cdots \to j_{1!}\mathbb{Q} \to j_{0!}\mathbb{Q} \to \mathbb{Q} \quad \text{is exact,} \tag{9}$$

and

$$H^j(M_i, \mathbb{Q}) = 0 \text{ for all } j \geq 1 \text{ and all } i \geq 0, \tag{10}$$

imply that $H^i(M, \mathbb{Q})$ is the i-th cohomology group of

$$0 \to H^0(M_0, \mathbb{Q}) \to H^0(M_1, \mathbb{Q}) \to \cdots \tag{11}$$

This follows by using Condition (9) in the first variable of $\mathrm{Ext}^*(\mathbb{Q}, \mathbb{Q})$ (which is $H^*(M, \mathbb{Q})$) to obtain a degenerate spectral sequence that degenerates to equation (11).

Next let us specialize to the case where there is a category, $\mathcal{U}$, and a topological space $M = M_{-1}$ with the following data. To each $X \in \mathrm{Ob}\,(\mathcal{U})$ there corresponds an *espace étalé*, $\iota_X \colon M_X \to M$. By an *correspondence étalé* from ι_X to ι_Y we mean *espaces étalés* $\eta_1 \colon N \to M_X$ and $\eta_2 \colon N \to M_Y$ such that $\iota_X \eta_1 = \iota_Y \eta_2$; we shall abbreviate this as $\eta \colon N \to M_X \times_M M_Y$ and speak of following η with projections $\mathrm{pr}_1, \mathrm{pr}_2$, respectively, to obtain η_1, η_2, respectively; since ι_X, ι_Y are *espaces étalés*, $M_X \times_M M_Y$ actually exists. Similarly we define an *r-correspondence étalé* on a tuple $\iota_{X_1}, \ldots, \iota_{X_r}$ via a map

$$\eta \colon N \to M_{X_1} \times_M \cdots \times_M M_{X_r}.$$

Definition 3.1 *Let $\mathcal{U}$ be a category and M a topological space. By a $\mathcal{U}$-covering of M, ι, we mean the data consisting of (1) for each $X \in \mathrm{Ob}(\mathcal{U})$ an* espace étalé *$\iota_X\colon M_X \to M$, and (2) for each $\phi \in \mathrm{Fl}(\mathcal{U})$ a correspondence étalé $\iota_\phi\colon M_\phi \to M_{s\phi} \times_M M_{t\phi}$.*

Consider a $\mathcal{U}$-covering of M, ι. For each composable sequence $\vec{\phi} = (\phi_1, \ldots, \phi_r)$ in $\mathcal{U}$, we can compose correspondences as usual to get an r-correspondence

$$\iota_{\vec{\phi}}\colon M_{\vec{\phi}} \to M_{s\phi_1} \times_M \cdots \times_M M_{s\phi_r} \times_M M_{t\phi_r}.$$

where

$$M_{\vec{\phi}} = M_{\phi_1} \times_{M_{t\phi_1}} \cdots \times_{M_{t\phi_{r-1}}} M_{\phi_r}.$$

We get a simplicial space étalé by setting

$$M_j = \coprod_{\phi \in \mathrm{Fl}^j(\mathcal{U})} M_\phi.$$

Definition 3.2 *We say that a $\mathcal{U}$-covering of M, ι, is* cohomologically faithful *if for each $j \geq 0$ and $\vec{\phi} \in \mathrm{Fl}^j(\mathcal{U})$ we have that $M_{\vec{\phi}}$ has the cohomology of a point (i.e., one dimensional in degree 0 and vanishing in higher degrees).*

In this case the cohomology of equation (11) is the cohomology of $\mathcal{U}$ (using the standard projective resolution of $\mathbb{Q}$ in $\mathcal{U}$). We have seen the following theorem.

Theorem 3.3 *Consider a cohomologically faithful $\mathcal{U}$-covering of M, ι, for which condition (9) holds. Then the cohomology of M is that of $\mathcal{U}$.*

To check condition (9) it suffices to check the stalks. Let $\mathcal{M}_p$ be the category whose objects are $j_0^{-1}(p)$ (recall j_i is the *espace étalé* $M_i \to M$) and morphisms are $j_1^{-1}(p)$ and source and target maps come from the correspondence. We have $j_i^{-1}(p)$ is the set of composable morphisms of length i in $\mathcal{M}_p$. The following theorem follows immediately.

Theorem 3.4 *Consider a $\mathcal{U}$-covering of M, ι. With notation as above, let each $\mathcal{M}_p$ be finite. Then condition (9) holds iff each $\mathcal{M}_p$ has the cohomology of a point. (Note that the empty category does not have the cohomology of a point.)*

We give some common practical situations. If each for each $\phi \in \mathrm{Fl}\,(\mathcal{U})$, ι_ϕ is a morphism $\iota_{s\phi} \to \iota_{t\phi}$, then $M_\phi = M_{s\phi}$ and we are in the following situation.

Theorem 3.5 *Consider a $\mathcal{U}$-covering of M, ι, for which the composition*

$$M_\phi \xrightarrow{\iota_\phi} M_{s\phi} \times_M M_{t\phi} \xrightarrow{\mathrm{pr}_1} M_{s\phi}$$

is an isomorphism. Then for $\vec{\phi} = (\phi_1, \ldots, \phi_k)$ we have $M_{\vec{\phi}} \simeq M_{s\phi_1}$. In particular, in such a situation we have that ι is cohomologically faithful provided that for each $X \in \mathrm{Ob}\,(\mathcal{U})$, M_X has the cohomology of a point.

Definition 3.6 *By a $\mathcal{U}$-*quasihypercover *of M, ι, we mean a $\mathcal{U}$-cover such that (1) for each $\phi \in \mathrm{Fl}\,(\mathcal{U})$, ι_ϕ is a morphism $\iota_{s\phi} \to \iota_{t\phi}$, (2) each $p \in M$ is in the image of some $\iota_X \colon M_X \to M$ for an $X \in \mathrm{Ob}\,(\mathcal{U})$, (3) all the ι_ϕ are distinct, (4) for each $X, Y \in \mathrm{Ob}\,(\mathcal{U})$, we have that $M_X \times_M M_Y$ is covered by the correspondences, meaning that it is the union of images $\iota_Z \to \iota_X \times \iota_Y$ of a product of two correspondences.*

Theorem 3.7 *Let $\mathcal{U}$ be a semitoplogical category (not necessarily finite). Let a topological space, M, admit a $\mathcal{U}$-quasihypercover, ι, such that each M_X has the cohomology of a point. Then the cohomology of M and $\mathcal{U}$ agree.*

Proof It suffices to show condition (9), i.e., that for any fixed p the sequence

$$\cdots \xrightarrow{d_1} \bigoplus_{\phi \in \mathrm{Fl}^1(\mathcal{M}_p)} \mathbb{Q} \xrightarrow{d_0} \bigoplus_{q \in \mathrm{Fl}^0(\mathcal{M}_p)} \mathbb{Q} \xrightarrow{d_{-1}} \mathbb{Q} \longrightarrow 0 \qquad (12)$$

is exact. Note that $\mathcal{M}_p$ is nonempty, by condition (2) of the definition of quasihypercover, and thus d_{-1} is surjective.

First we claim that each $\mathcal{M}_p$ is of topological type. Indeed, assume not. Then there exist μ_1, μ_2, distinct morphisms $\iota_X \to \iota_Y$ for objects $X, Y \in \mathrm{Ob}\,(\mathcal{U})$, such that for $q \in M_X$ with $\iota_X(q) = p$ we have $\mu_1(q) = \mu_2(q)$. The set where $\mu_1 = \mu_2$ is closed (in any topological setting) and open (since they are *espaces étalés*), contains a point (namely, q), but is not all of M_X; therefore M_X has at least two connected components and does not have the cohomology of a point.

For each $q_1, q_2 \in \mathrm{Ob}\,(\mathcal{M}_p)$, condition (4) on quasihypercovers shows that there is a $q_3 \in \mathrm{Ob}\,(\mathcal{M}_p)$ with arrows to both q_1 and q_2. In particular, if $\mathcal{M}_p$ is finite then it has an initial element.

Assume that $\mathcal{M}_p$ is finite. Any category of topological type with an initial element has the cohomology of a point (using the fact that $\mathbb{Q}$ is injective, equalling $j_*\mathbb{Q}$ where j is the inclusion of the initial element of the category into the category). So we are done.

If $\mathcal{M}_p$ is not finite, any element, η, of a direct sum in equation (12) vanishes on all but finitely many components, and is therefore supported on a finite, full subcategory $\mathcal{M}'_p$ of $\mathcal{M}_p$. If $\mathcal{M}'_p$ has k objects, $X_1, \ldots, X_k$, set $Y_1 = X_1$ and let Y_i be inductively defined for $i \geq 2$ as an element with an arrow to X_i and Y_{i-1}. Let $\mathcal{M}''_p$ be the full subcategory of $\mathcal{M}_p$ on the objects $X_1, Y_1, \ldots, X_k, Y_k$. The $\mathcal{M}''_p$ is finite, has an initial element (namely Y_k), and supports η; as mentioned before, this means $\mathcal{M}''_p$ has the cohomology of a point. Hence if $\eta \in \ker d_i$, then it is in the image of d_{i+1} (first restricting to $\mathcal{M}''_p$ and then extending by zero to $\mathcal{M}_p$).

□

We remark that this theorem does not cover all interesting cases. The following class of examples is joint with Denis Sjerve. Consider again $M = M_{-1} = \mathbb{R}/\mathbb{Z}$, $M_0 = U_1 \amalg U_2$ with $U_2 = (-.1, 3.1)$ and $U_1 = (-.1, 2.1)$ mapped naturally to M. We set M_1 to be two copies of U_1, representing the two maps addition by 0 and by 1. $U_2 \times_M U_2$ consists of a number of "strips," i.e., pairs (a, b) in $U_2 \times U_2$ with $a - b \in \mathbb{Z}$, but only the longer strips are "covered" by U_1 (that covers $a - b = \pm 1$) and U_2 (that covers $a = b$). So this is not a quasihypercover. On the other hand, each $\mathcal{M}_p$ has the cohomology of a point, as do the U_i's, and so the cohomology of the resulting category is that of M.

(More interesting examples can be obtained with $\mathbb{R}/\mathbb{Z}$ covered by more and different size intervals.)

3.3 Examples

Here we give some examples of categories.

Consider the boundary of a simplex on n-vertices, and extend each of its n faces (of dimension $n-1$) slightly to an open set. The resulting category is $(\Delta_1)^n$ minus its initial object and its terminal object; this category and morphism therefore models the sphere S^{n-1}. (This corresponds to a good cover; the quasihypercover consists of the set of intersections of any of the extended faces.)

Cover the sphere S^n with an open upper and lower "extended" hemispheres (each extending past the "equator"), covering the hemisphere intersection $\simeq \mathbb{R} \times S^{n-1}$ by covering S^{n-1} with extended hemispheres, etc., we see that S^n admits a quasihypercover associated to a category whose objects are $\{U_0, L_0, \ldots, U_n, L_n\}$ with inclusions as the object index is increased. (This is comes from a pretty good cover, where the intersection of hemispheres can have nontrivial cohomology.)

As mentioned before, a line segment that meets itself at its ends gives rise to a category with objects $\{M_0, M_1\}$, with two morphisms from M_0 to M_1, corresponding to the two inclusions of the self-intersection segment in the circle into the line segment. Another way to achieve this is to act on the category in the previous paragraph by the cyclic group of order two, corresponding to the antipodal map on the sphere. We conclude that real projective n-space (the quotient of S^n by the antipodal map) is modelled by the category with objects $\{M_0, \ldots, M_n\}$, with two morphisms between objects of increasing index which can be labelled $\{+, -\}$ such that composition given by multiplying signs.

We finish with some general (less geometric) remarks on finite categories. Δ_1 can be viewed as the tautological open/closed pair, in that to given an open (or closed) set in $\mathcal{C}$ is the same as to give a morphism $\mathcal{C} \to \Delta_1$. Similarly $(\Delta_1)^n$ is the tautological ordered n-tuple of open/closed pairs. New categories can be obtained from old ones by limits; we shall be especially interested in the fiber product.

An"$(m+1)$-partite" category is a category of topological type, $\mathcal{C}$, with objects consisting of $m+1$ sets $S_0, \ldots, S_m$, with $|S_i| = n_i$, and with respectively one or zero morphism from an object in S_i to one in S_j according to whether or not $i \leq j$. The greedy resolution easily shows that

$$h^i(\mathcal{C}, \mathbb{Q}) = \begin{cases} 1 & \text{if } i = 0, \\ (n_0 - 1) \ldots (n_m - 1) & \text{if } i = m, \\ 0 & \text{otherwise.} \end{cases}$$

The standard resolution shows that the m-th Betti number of $\mathcal{C}$ can be bounded by the number of composable m-tuples not containing an identity. The above computation for $\mathcal{C}$ shows that this bound can be "close" to true (at least for m fixed and all n_i "large"), since the number of such tuples is $n_0 \ldots n_m$.

4 Virtual Zero Extensions

Consider equation (1). Assume that $U_{f\wedge g} = U_f \cap U_g$ for all f, g. Let Z_g be the closed complement to U_g, for each g. The exact sequence

$$0 \to G_{U_g} \to G \to G_{Z_g} \to 0,$$

shows that $\mathrm{cc}(F, G_{Z_g})$ is within $\mathrm{cc}(F, G)$ of $\mathrm{cc}(F, G_{U_g})$.

Definition 4.1 *Let $G \in \mathbb{Q}(\mathcal{C})$. Let U, Z respectively be an open and a closed set in $\mathcal{C}$. A* virtual $G_{U,Z}$ *(or* virtual zero extension for G, U, Z*) is a sheaf H and arrows $G_U \to H \to G_Z$ such that*

$$0 \to G_U \to G_{U\cap Z} \oplus H \to G_Z \to 0;$$

is exact, where the map to $G_{U\cap Z}$ is the identity on $U \cap Z$ and the map from $G_{U\cap Z}$ is minus the identity on $U \cap Z$.

Virtual $G_{U,Z}$'s form a category, with a morphism from H_1 to H_2 (with arrows from G_U and to G_Z) being $\mu\colon H_1 \to H_2$ such that

$$\begin{array}{ccccc} G_U & \longrightarrow & H_1 & \longrightarrow & G_Z \\ \big\downarrow{\scriptstyle \mathrm{Id}} & & \big\downarrow{\scriptstyle \mu} & & {\scriptstyle \mathrm{Id}}\big\downarrow \\ G_U & \longrightarrow & H_2 & \longrightarrow & G_Z \end{array}$$

commutes everywhere. Notice that for G as above and any full subcategory, A, of $\mathcal{C}$, we can define G_A, the "literal extension of G on A by 0," to be the sheaf that agrees with G on A and is extended by 0 outside of G if this sheaf exists; the issue in existence is that if $\phi \in \mathrm{Fl}(\mathcal{C})$ factors through an element outside of A, then $G_A\phi$ is forced to 0, which creates a conflict if $G\phi$ is not zero. If G_A exists and $A = U \cup Z$ with U open, Z closed, then G_A is a virtual $G_{U,Z}$.

The following theorem is easy to check.

Theorem 4.2 *Let $\mathcal{C}$ be a finite category of topological type, and $A \subset \mathrm{Ob}(\mathcal{C})$. The following are equivalent:*

1. *A is an* open/closed intersection*, i.e., the intersection of an open set with a closed set,*

2. *A is the intersection of* open(A) *with* closed(A), *where* open(A) *is the smallest open set containing A and similarly for* closed(A),

3. *A is* cavity-free, *i.e., if ϕ_1, ϕ_2 are composable morphisms with $s\phi_1, t\phi_2 \in A$, then $t\phi_1 = s\phi_2 \in A$,*

4. *for all $F \in \mathbb{Q}(\mathcal{C})$ we have that the literal zero extension F_A exists.*

Notice that for $\mathcal{C} = \Delta_2$ and $A = 0, 2$, the conditions of the theorem do not hold, and yet a virtual $F_{\{0\},\{2\}}$ always exists.

If $U \cap Z = \emptyset$ (with U open, Z closed), then the category of virtual $G_{U,Z}$'s is the same as the Yoneda $\mathrm{Ext}(G_Z, G_U)$ category.

Virtual extensions always exist in free categories (that don't have the type of conflict described earlier, since each morphism has a unique factorization).

Let us make some structural observations.

Definition 4.3 *A virtual $G_{U,Z}$, H, is* standard *if (1) $H(X)$ is $G(X)$ or 0 according to whether or not $X \in U \cup Z$, (2) $H\phi = 0$ if $s\phi$ or $t\phi$ lies outside $U \cup Z$, (3) $H\phi = G\phi$ if $s\phi$ and $t\phi$ both lie in U or both in Z.*

For a standard virtual $G_{U,Z}$, H, the only morphisms $H\phi$ that are not determined are those with $s\phi \in U \setminus Z$ and $t\phi \in Z \setminus U$.

Theorem 4.4 *In the category of virtual $G_{U,Z}$'s, each isomorphism class contains exactly one standard virtual $G_{U,Z}$. The virtual $G_{U,Z}$'s form a $\mathbb{Q}$-vector space, via (a modification of) the Baer sum; when we restrict this sum to standard virtual $G_{U,Z}$'s, the vector space structure is given by mapping H to the sum indexed over $\phi \in \mathrm{Hom}(X, Y)$ with $Y \in Z \setminus U$ and $X \in U \setminus Z$ of $H\phi \in \mathrm{Hom}(G(Y), G(X))$ (and we may restrict to ϕ prime if we like).*

Proof If for each $X \in \mathrm{Ob}(\mathcal{C})$ we have isomorphisms $\iota_X : H(X) \to t\iota_X$, then we define the *conjugate of H by ι* to be the sheaf, H', such that $H'(X) = t\iota_X$ and $H'\phi = \iota_{s\phi}(H\phi)\iota_{t\phi}^{-1}$. Note that H' is in the same isomorphism class as H, with ι giving rise to an isomorphism $H \to H'$.

The exact sequence

$$0 \to G_U \to G_{U\cap Z} \oplus H \to G_Z \to 0$$

shows that for $X \in U \setminus Z$ we may choose an isomorphism $\iota_X : H(X) \to G(X)$. Similarly for $X \in Z \setminus U$. For $X \in U \cap Z$, we have

$$0 \to G(X) \xrightarrow{\mathrm{Id}\oplus\alpha} G(X) \oplus H(X) \xrightarrow{(-\mathrm{Id})\oplus\beta} G(X) \to 0.$$

We have $\beta\alpha = \mathrm{Id}$; set $\iota_X = \alpha$. Then we easily check that H conjugated by ι gives rise to a standard virtual $G_{U,Z}$.

If $G_U \xrightarrow{\alpha_i} H_i \xrightarrow{\beta_i} G_Z$ for $i = 1, 2$ are two virtual $G_{U,Z}$'s, for each $X \in \mathrm{Ob}\,(\mathcal{C})$ we consider pairs (h_1, h_2), with $h_i \in H_i(X)$ such that $\beta_1(h_1) - \beta_2(h_2)$ agree on $Z \backslash U$; let $H_3(X)$ be the set of such pairs modulo the image of $\alpha_1 \oplus \alpha_2$. (This construction comes from the Baer sum, $G_{U\cap Z} \oplus H_i$ being an extension of G_Z by G_U.) We easily check H_3 is a virtual $G_{U,Z}$. Furthermore, if H_1, H_2 are standard, then so is H_3, and for ϕ with $t\phi \in Z \setminus U$ and $s\phi \in U \setminus Z$ we have

$$H_3\phi = H_1\phi + H_2\phi.$$

Finally, the $H\phi$ as above are determined by $H\phi$ for ϕ prime, since any factorization of such a ϕ contains exactly one morphism with source in $U \setminus Z$ and target in $Z \setminus U$.

$\square$

Assume for each f, g there is a virtual G_{U_f,Z_g}. Then the resulting short exact sequence gives

$$\mathrm{cc}(F, G_{U_f\cap Z_g}) \le \mathrm{cc}(G_{U_f}) + \mathrm{cc}(G_{Z_g}).$$

The exact sequence

$$0 \to G_{U_f\cap U_g} \to G_{U_f} \to G_{U_f\cap Z_g} \to 0,$$

gives

$$\mathrm{cc}(F, G_{U_f\cap U_g}) \le \mathrm{cc}(G_{U_f}) + \mathrm{cc}(G_{U_f\cap Z_g}).$$

We conclude

$$\mathrm{cc}(f \wedge g) \le 2\ \mathrm{cc}(f) + \mathrm{cc}(g) + \mathrm{cc}(F, G).$$

Virtual zero extensions exist in the following two extreme cases: (1) each U_f is also closed, and (2) $\mathcal{C}$ is a free category. The problem with the first case is all the U_f, Z_f's are disconnected, and the bounds are trivial. The problem with the second is that $\widehat{\mathcal{C}}$ is homologically one dimensional (see Subsection 2.11), and we think it less likely that sheaf models based on such $\mathcal{C}$ will give interesting bounds. For example, all cohomology boils down to H^0 and the Euler characteristic; H^0 is usually simple to determine, and the Euler characteristic has the simple formula:

$$\chi(G) = \sum_{P\in\mathrm{Ob}(\mathcal{C})} \dim\bigl(G(P)\bigr) - \sum_{\phi\ \text{prime}} \dim\Bigl(G\bigl(\mathrm{source}(\phi)\bigr)\Bigr).$$

It is possible to give some simple variants on these ideas. For example, one could find a "near zero extension," i.e., an H such that the derivation of (the non-exact in the middle):

$$0 \to G_U \to G_{U\cap Z} \oplus H \to G_Z \to 0$$

is not nonzero, but has, say, a middle term, M; then we simply add $\mathrm{cc}(F, M)$ appropriately into the bounds.

Let us note that virtual zero extensions are preserved under pulling back. In other words, if $u\colon \mathcal{C}' \to \mathcal{C}$ is an arbitrary functor, and we have that H is a virtual $G_{U,Z}$ (all, as before, over $\mathcal{C}$), then u^*H is a virtual $(u^*G)_{u^{-1}(U),u^{-1}(Z)}$. Furthermore, say that there is a virtual $G_{U,Z}$ *in* $\mathcal{C}'$ (with $u\colon \mathcal{C}' \to \mathcal{C}$ understood, or say *in* u) if there is a virtual $(u^*G)_{u^{-1}(U),u^{-1}(Z)}$. So each time we pullback to a category, vitual zero extensions are never destroyed and new ones may be created (although the notion of the zero extension depends, of course, on how the sheaf and open and closed sets pullback).

This suggests a possible fiber product construction. Say that for each f, g we choose a $u = u_{f,g}\colon \mathcal{C}_{f,g} \to \mathcal{C}$ such that if there is a virtual G_{U_f,Z_g} in $\mathcal{C}_{f,g}$. The fiber product, $\mathcal{X}$, of the $\mathcal{C}_{f,g}$'s over $\mathcal{C}$ has a virtual G_{U_f,Z_g} in $\mathcal{X}$ for any f, g. Of course, such a construction, given that there are 2^{2^n} possible f's and possible g's, would yield a large category.

It is for this reason that we study the behavior of duality (which we have in mind for negation) under fiber products (and therefore, more generally, under arbitrary base change).

In the sections to follow we will see that it may be possible to maintain a reasonable duality theory while performing fiber product operations.

5 Injectives and Projectives

In this section we describe the structure of injective and projective modules in $\mathbb{Q}(\mathcal{C})$ for a finite, semitopological category, $\mathcal{C}$. Then we describe a naturally arising map $* \to !$ and its quasi-inverse $! \to *$ (defined in the derived category); intuitively, $* \to !$ is constructed by taking a complex of sheaves, writing an injective resolution, writing each injective as a sum of modules $(k_X)_*V$ (see below), and replacing the $*$ with a !. Then we define the trace of a map, either from an injective or projective to itself, or from an injective, I, to $I^{*\to !}$, or from a projective, P, to $P^{!\to *}$; this trace is constructed using the structure of injectives and projectives.

Let $\mathcal{C}$ be a finite semitopological category, and let $\mathcal{I}$ be the injectives of $\mathbb{Q}(\mathcal{C})$. We begin by describing $\mathcal{I}$.

If $X \in \mathrm{Ob}(\mathcal{C})$, we set $k_X\colon \Delta_0 \to \mathcal{C}$ to be the inclusion functor mapping 0 to X. Let $\mathcal{V} = \mathbb{Q}(\Delta_0)$ be the category of vector spaces. For $V \in \mathcal{V}$ we have $(k_X)_* V$ is injective.

Theorem 5.1 *Every element of $\mathcal{I}$ is the direct sum of injectives of the form $(k_X)_*V$. More precisely, an $I \in \mathcal{I}$ is the sum of $(k_X)_*V_X$, where for each X, if $\phi_1, \ldots, \phi_r$ are the morphisms with target X, then*

$$V_X = \ker(I\phi_1 \oplus \cdots \oplus I\phi_r). \tag{13}$$

Proof We prove the theorem by induction on the number of objects in $\mathcal{C}$. Let X be an object such that $I(X)$ is nonzero and X is minimal with this property, i.e., if ϕ is a morphism with target X then I is zero on the source of ϕ. Let $k = k_X$, $V = I(X)$, and $G = k_*V$; let $G_X = k_!V$ be the sheaf that is zero outside X and with $G(X) = V$. The inclusion of G_X into G and the map from G_X to I gives rise to a map $\psi\colon G \to I$.

We claim that for any Y we have $\psi(Y)\colon G(Y) \to I(Y)$ is an injection (and therefore ψ is an injection). Indeed, let $\phi_1, \ldots, \phi_s$ be the morphisms from X to Y. We have a commuting diagram

$$\begin{array}{ccc} G(Y) & \xrightarrow{G\phi_1 \oplus \cdots \oplus G\phi_s} & G(X)^s \\ \downarrow & & \downarrow \\ I(Y) & \xrightarrow{I\phi_1 \oplus \cdots \oplus I\phi_s} & I(X)^s \end{array}$$

The top arrow is an isomorphism, as is the right arrow. Hence the left arrow is an injection, which was the claim.

Hence $\psi\colon G \to I$ is an injection. A standard argument shows that the cokernel of an injection of injectives is injective (see, e.g., [GM03]), and thus the cokernel I/G is also injective, and hence I is a direct sum of G and I/G. Thus equation (13) holds (for that particular X). Let $\mathcal{C}'$ be the full subcategory of $\mathcal{C}$ with X removed. I/G vanishes at X, and so I/G's restriction to $\mathcal{C}'$ is injective; by induction we have that I/G restricted to $\mathcal{C}'$ is a direct sum as above; the same is true viewing I/G on $\mathcal{C}$ (extended by 0 on X). For any Y, let $\phi_1, \ldots, \phi_s$ be the morphisms from X to Y. Then $I\phi_1 \oplus \cdots \oplus I\phi_s$ is an injection on $G(Y)$ and vanishes on $(I/G)(Y)$. This shows equation (13) with (arbitrary) Y replacing X, given that it holds for I/G on $\mathcal{C}'$.

□

Let $\mathcal{C}$ be a finite semitopological category and $\mathcal{V}$ be the category of all finite dimensional vector spaces, with notation as before. Let $\nu\colon \mathrm{SHom}\,(\mathcal{C},\mathcal{V}) \to \mathcal{I}$ be the functor given as follows; for $F \in \mathrm{Hom}\big(\mathrm{Ob}\,(\mathcal{C}),\mathrm{Ob}\,(\mathcal{V})\big)$ set

$$\nu(F) = \bigoplus_{X\in\mathrm{Ob}(\mathcal{C})} (k_X)_* F(X).$$

Notice that

$$\mathrm{Hom}\big((k_X)_*F(X),(k_Y)_*G(Y)\big) \simeq \mathrm{Hom}\big(k_Y^*(k_X)_*F(X),G(Y)\big)$$

$$\simeq \bigoplus_{\phi\colon X\to Y} \mathrm{Hom}\big(F(X),G(Y)\big);$$

A morphism, $H\colon F \to G$, in $\mathrm{SHom}\,(\mathcal{C},\mathcal{V})$ gives for each $\phi\colon X \to Y$ an element of $\mathrm{Hom}\big(F(X),G(Y)\big)$, and therefore an element of

$$\mathrm{Hom}\big((k_X)_*F(X),(k_Y)_*G(Y)\big)$$

for each X and Y, and therefore a morphism in $\mathcal{I}$. We easily verify that this makes ν a functor.

Theorem 5.2 *The functor ν defines an equivalence of categories between* $\mathrm{SHom}\,(\mathcal{C},\mathcal{V})$ *and $\mathcal{I}$; in other words, ν is fully faithful and essentially surjective.*

Proof Theorem 5.1 shows that ν is essentially surjective. Fix objects F, G of $\mathrm{SHom}\,(\mathcal{C},\mathcal{V})$. A morphism from a direct sum to another direct sum decomposes into morphisms from each direct summand to each in the other. Thus there is a one-to-one correspondence between elements of $\mathrm{Hom}(\nu F,\nu G)$ and the direct sum over X, Y objects of $\mathcal{C}$ of

$$\mathrm{Hom}\big((k_X)_*F(X),(k_Y)_*G(Y)\big) \simeq \bigoplus_{\phi\colon X\to Y} \mathrm{Hom}\big(F(X),G(Y)\big).$$

But the right-hand-side summed over all X, Y just gives $\mathrm{Hom}(F,G)$; this gives a one-to-one correspondence between $\mathrm{Hom}(F,G)$ and $\mathrm{Hom}(\nu F,\nu G))$. Hence ν is fully faithful.

□

Henceforth with denote by $\nu_{\to *}$ the functor ν in the above theorem.

Now we wish to describe a certain class of quasi-inverses to $\nu_{\to *}$ that we will use. Given $I \in \mathrm{Ob}\,(\mathcal{I})$ we define

$$(\mu I)(X) = \ker \left(\bigoplus_{t(\phi)=X} I\phi \right)$$

(as in equation (13)). It is easy to see that for each $F \in \mathrm{Ob}\,(\mathrm{SHom}\,(\mathcal{C},\mathcal{V}))$, for each $X \in \mathrm{Ob}\,(\mathcal{C})$ we have that $F(X) \simeq \mu\nu_{\to *}F(X)$. In a sense μ comes close to being a quasi-inverse; an $f\colon I_1 \to I_2$ clearly determines a map

$$(\mu f)(\mathrm{Id}(X))\colon (\mu I_1)(X) \to (\mu I_2)(X).$$

Thus we may speak of μ as a map on objects and "diagonal parts of morphisms." However, given $\phi \in \mathrm{Fl}\,(\mathcal{C})$ with $\phi\colon X \to Y$ and $X \neq Y$, there seems to be no canonical choice for a morphism

$$(\mu f)\phi\colon (\mu I_1)(X) \to (\mu I_2)(Y).$$

(For example, try to make a canonical choice in the case where $\mathcal{C} = \Delta_1$, $X = 0$, $Y = 1$, and I_1, I_2 are both isomorphic to J, where $J(0) \simeq \mathbb{Q}$ and $J(1) \simeq \mathbb{Q}^2$ (therefore $J(1) \to J(0)$ is surjective, since J is injective); how can an $f\colon I_1 \to I_2$ determine a map from $(\mu I_1)(0)$ to $(\mu I_2)(1)$?)

Definition 5.3 *By the* "to star" functor *we mean the functor $\nu_{\to *}$ above. By the* "from star" functor*, $\nu_{*\to}$ we mean any quasi-inverse that agrees with μ as a map on objects and on diagonal parts of morphisms.*

In other words, for each $I \in \mathcal{I}$ we choose an isomorphism $\iota\colon \nu_{\to *}\mu I \to I$ such that μ maps ι to the identity map on the diagonal parts of ι; such an ι exists by the fully faithfulness of $\nu_{\to *}$. Then the choice of such an ι for each I determines, as usual, a quasi-inverse $\nu_{*\to}$.

Note that the Axiom of Choice implicit in the last paragraph is not really necessary if we are interested in applying it (in practice) to only finitely many objects (see Section 2.9.1).

Let us mention that our restriction to a special type of quasi-inverse, $\nu_{*\to}$, will make the definition of a certain trace independent of the choice of quasi-inverse; see below. (However, it is not clear to us that this independence is absolutely necessary.)

By duality, and in particular by replacing $*$ with !, we get a similar functor (called "to shriek") $\nu_{\to !}\colon \mathrm{SHom}\,(\mathcal{C},\mathcal{V}) \to \mathcal{P}$ where $\mathcal{P}$ is the category of projectives of $\mathbb{Q}\,(\mathcal{C})$, with quasi-inverse ("from shriek") $\nu_{!\to}$. Let $! \to *$ denote the functor $\nu_{\to *} \circ \nu_{!\to}$, and similarly for $* \to !$ (exchanging $*$ and ! in the subscripts). These functors are clearly additive, and therefore give rise to functors from $\mathcal{K}(\mathcal{P})$ to $\mathcal{K}(\mathcal{I})$ and back (that are quasi-inverses). They therefore give rise to δ-functors on $\mathcal{D}^{\mathrm{b}}(\mathbb{Q}\,(\mathcal{C}))$. (Again, the Axiom of Choice is used to construct quasi-inverses of the natural maps from $\mathcal{K}^{\mathrm{b}}(\mathcal{P})$ and $\mathcal{K}^{\mathrm{b}}(\mathcal{I})$ to $\mathcal{D}^{\mathrm{b}}(\mathbb{Q}\,(\mathcal{C}))$; this conceptually simplifying use of the Axiom of Choice can be avoided as discussed in Section 2.9.1.) We alternatively denote $(* \to !)F$ by $F^{*\to !}$ and similarly for $! \to *$.

We note that, using Section 2.9.3, any two $! \to *$ constructed (from two different "from shriek" quasi-inverses $\nu_{!\to}$) are isomorphic. Similarly for $* \to !$.

Let $F \in \mathrm{SHom}\,(\mathcal{C},\mathcal{V})$, and let $f \in \mathrm{Hom}(F,F)$. We define the trace of f to be

$$\mathrm{Tr}(f) = \sum_{X\in \mathrm{Ob}(\mathcal{C})} \mathrm{Tr}\big(f(\mathrm{Id}_X)\big).$$

where

$$f(\mathrm{Id}_X)\colon F(X) \to F(X)$$

is the restriction of f to the identity morphism on X, which is a linear map from $F(X)$ to itself and therefore has a trace.

We claim that this trace is invariant under conjugacy, i.e., that if $\iota\colon F \to G$ is an isomorphism in $\mathrm{SHom}\,(\mathcal{C},\mathcal{V})$, then $\mathrm{Tr}(\iota f \iota^{-1}) = \mathrm{Tr}(f)$. To see this first note that for any $\delta \in \mathrm{Fl}\,(\mathcal{C})$ we have

$$\sum_{\gamma\alpha=\delta} \iota^{-1}(\gamma)\iota(\alpha) = \begin{cases} \mathrm{Id}_{F(X)} & \text{if } \delta = \mathrm{Id}_X \text{ for some } X, \\ 0 & \text{otherwise,} \end{cases}$$

by definition of composition and since $\iota^{-1}\iota = \mathrm{Id}_{s\iota}$. So

$$\sum_X \mathrm{Tr}\big((\iota f \iota^{-1})(\mathrm{Id}_X)\big) = \sum_{X,\alpha\beta\gamma=\mathrm{Id}_X} \mathrm{Tr}\big(\iota(\alpha)f(\beta)\iota^{-1}(\gamma)\big)$$

which, since $\alpha\beta\gamma = \mathrm{Id}_{t\alpha}$ iff $\beta\gamma\alpha = \mathrm{Id}_{t\beta}$,

$$= \sum_{X,\beta\gamma\alpha=\mathrm{Id}_X} \mathrm{Tr}\big(f(\beta)\iota^{-1}(\gamma)\iota(\alpha)\big) = \sum_X \mathrm{Tr}\big(f(\mathrm{Id}_X)\big).$$

Thus $\mathrm{Tr}(\iota f \iota^{-1}) = \mathrm{Tr}(f)$.

Next we extend the trace on $\mathrm{Hom}(I, I)$ for each $I \in \mathcal{I}$. To do this we choose an $F \in \mathrm{SHom}\,(\mathcal{C}, \mathcal{V})$ and an $\iota \in \mathrm{Fl}\,(\mathcal{I})$ such that $\iota\colon \nu_{\to *}(F) \to I$ is an isomorphism. Given $f \in \mathrm{Hom}(I, I)$, $\nu_{\to *}$ sets up a one-to-one correspondence between $\mathrm{Hom}(F, F)$ and $\mathrm{Hom}(\nu_{\to *}F, \nu_{\to *}F)$, and if g is mapped to $\iota f \iota^{-1}$ we define

$$\mathrm{Tr}(f) = \mathrm{Tr}(g).$$

We claim this definition of trace is independent of the choice of F and ι. Indeed, let F', ι' be another such choice. Then F, F' are conjugate under the morphism that maps (under $\nu_{\to *}$) to $\iota^{-1}\iota'$ and the g' that maps to $\iota' f' (\iota'^{-1})$ is conjugate to g under this map. Therefore $\mathrm{Tr}(g) = \mathrm{Tr}(g')$.

Next consider $f \in \mathrm{Hom}(I^{*\to !}, I)$ for $I \in \mathrm{Ob}\,(\mathcal{I})$. We have $\nu_{*\to} I = \mu I$; also $\nu_{*\to}$ on morphisms involving I is determined by the choice of an $\iota\colon \nu_{\to *}\mu I \to I$. We get

$$\iota^{-1} f \in \mathrm{Hom}(\nu_{\to !}(F), \nu_{\to *}(F)).$$

From the direct sum decompositions of $\nu_{\to !}(F), \nu_{\to *}(F)$ we get restrictions, for each $X \in \mathrm{Ob}\,(\mathcal{C})$

$$(\iota^{-1} f)|_{\mathrm{Id}(X)}\colon (k_X)_! F(X) \to (k_X)_* F(X).$$

Since plainly

$$\mathrm{Hom}\big((k_X)_! V, (k_X)_* V\big) \simeq \mathrm{Hom}(V, V)$$

on which we have the trace defined, we can define

$$\mathrm{Tr}(\iota^{-1} f) = \sum_{X \in \mathrm{Ob}(\mathcal{C})} \mathrm{Tr}\left((\iota^{-1} f)|_{\mathrm{Id}(X)}\right).$$

But $\iota^{-1} f$ on $\mathrm{Id}(X)$ is independent of ι (since the ι must agree with μ on diagonal parts of morphisms). Thus this trace is independent of the choice of ι and we can unabmiguously denote it $\mathrm{Tr}(f)$.

If $f \in \mathrm{Hom}(P, P^{!\to *})$ for P projective, we can similarly define the trace of f.

6 Ext Duality

Let $\mathcal{C}$ be a finite semitopological category. Let $\mathcal{D} = \mathcal{D}^{\mathrm{b}}(\mathbb{Q}(\mathcal{C}))$ be, as usual, the derived category of bounded $\mathbb{Q}(\mathcal{C})$ complexes. For $G \in \mathcal{D}$ we define a functor, $G^{\mathrm{L}\to\mathrm{R}}$,

$$F \mapsto \big(\mathrm{Hom}_{\mathcal{D}}(G, F)\big)^*.$$

In this section we prove the following theorem.

Theorem 6.1 *For every G, $G^{\mathrm{L}\to\mathrm{R}}$ is representable by $G^{!\to *}$.*

We call $G^{\mathrm{L}\to\mathrm{R}}$ a the *left to right Ext dual of* G, for the following reason.

Corollary 6.2 *Let $G^{!\to *} \simeq H[n]$ in $\mathcal{D}$, where $G, H \in \mathbb{Q}(\mathcal{C})$. Then for each $F \in \mathbb{Q}(\mathcal{C})$ we have*

$$\mathrm{Ext}(F, H) = \mathrm{Ext}^n(G, F).$$

In the case of the corollary above we say that H is the *n-dimensional left to right Ext dual of F*. As mentioned before, $\mathrm{L} \to \mathrm{R}$ or $! \to *$ is also known as the *Serre functor*.

Corollary 6.3 *If $F^{!\to *} = F[n]$ and $G^{!\to *} = G'[n']$ for sheaves $G, G' \in \mathbb{Q}(\mathcal{C})$ and integers n, n' then*

$$\mathrm{cc}(F, G) = \mathrm{cc}(F, G').$$

The corollary follows since

$$\dim\bigl(\mathrm{Ext}^i(F, G)\bigr) = \dim\bigl(\mathrm{Hom}_{\mathcal{D}}(F, G[i])\bigr)$$

$$= \dim\bigl(\mathrm{Hom}_{\mathcal{D}}(G[i], F[n])\bigr) = \dim\bigl(\mathrm{Hom}_{\mathcal{D}}(F[n], G'[i + n'])\bigr).$$

We finish this subsection with the proof of Theorem 6.1.

First, if $A \in \mathcal{D}([0])$ we easily see that

$$\mathrm{Hom}_{\mathcal{K}}(\mathbb{Q}, A) \simeq H^0(A),$$

and

$$\mathrm{Hom}_{\mathcal{K}}(A, \mathbb{Q}) \simeq \bigl(H^0(A)\bigr)^*.$$

Next for $A \in \mathrm{Ob}(\mathcal{C})$, consider $k_A \colon [0] \to \mathcal{C}$ as before. Let $B \in \mathcal{D}^{\mathrm{b}}(\mathbb{Q}(\mathcal{C}))$. Then, using the fact that $(k_A)_!\mathbb{Q}$ is projective,

$$\mathrm{Hom}_{\mathcal{D}}((k_A)_!\mathbb{Q}, B) \simeq \mathrm{Hom}_{\mathcal{K}(\mathcal{C})}((k_A)_!\mathbb{Q}, B) \simeq \mathrm{Hom}_{\mathcal{K}([0])}(\mathbb{Q}, k_A^* B) \simeq H^0(k_A^* B),$$

and similarly

$$\mathrm{Hom}_{\mathcal{D}}(B, (k_A)_*\mathbb{Q}) \simeq \bigl(H^0(k_A^* B)\bigr)^*.$$

This shows that for *fixed* A, the functor $((k_A)_!\mathbb{Q})^{\mathrm{L}\to\mathrm{R}}$ is represented by $(k_A)_*\mathbb{Q}$.

Definition 6.4 *Consider a functor $F\colon \mathcal{T} \to \mathbb{Q}(\mathcal{A})$, where $\mathcal{T}$ is a triangulated category and $\mathcal{A}$ is an additive category. We say that F is a* weak δ-functor *if F is additive and for every distinguished triangle, $X \to Y \to Z \to X[1]$ in $\mathcal{T}$ and every $A \in \mathcal{A}$ we have*

$$\cdots \to \mathrm{Hom}(FX[i], A) \to \mathrm{Hom}(FY[i], A) \to$$

$$\mathrm{Hom}(FZ[i], A) \to \mathrm{Hom}(FX[i+1], A) \to \cdots$$

is exact.

For example, let $\mathcal{A}$ be a triangulated category and $F'\colon \mathcal{T} \to \mathcal{A}$ a δ-functor. Then F' followed by the (vector space) Yoneda embedding is clearly a weak δ-functor.

Definition 6.5 *A subset of objects, I, of a triangulated category, $\mathcal{T}$, is* triangularly closed *if for each distinguished triangle $T_1 \to T_2 \to T_3 \to T_1[1]$ in $\mathcal{T}$, if any two of T_1, T_2, T_3 lie in I, then so does the third. The* triangular closure *of a set of objects, I, in $\mathcal{T}$ is the intersection of all triangularly closed sets in $\mathcal{T}$; we say I* triangularly generates $\mathcal{T}$ *if its triangular closure consists of all objects in $\mathcal{T}$.*

Theorem 6.6 *Let F, G be two weak δ-functors from $\mathcal{T}$ to $\mathbb{Q}(\mathcal{A})$, and let $u\colon F \to G$ be a morphism of functors. Then the set of objects of $\mathcal{T}$ on which u is an isomorphism is triangularly closed.*

Proof This follows from the five-lemma (compare [Har66], Proposition I.7.1).

□

Theorem 6.7 *Let T be a subset of objects of an abelian category, $\mathcal{A}$, such that every object of A has a finite resolution whose objects are finite sums of elements in T. Then T triangularly generates $\mathcal{D}^{\mathrm{b}}(\mathcal{A})$.*

Proof This follows from [Har66], Lemma I.7.2 (a distinguished triangle in $\mathcal{D}(\mathcal{A})$ consisting of an arbitrary middle element and "truncations" from above and below on either side; also called "filtrations" in [GM03], III.7.5), and from the fact that if $A, B \in \mathcal{A}$ then there is a distinguished triangle

$$A \to A \oplus B \to B \to A[1]$$

□

Next we wish to define a functor, u, from $! \to *$ to $\mathrm{L} \to \mathrm{R}$, and to show (1) u is an isomorphism on each $(k_A)_*$, and then conclude (2) therefore u is an isomorphism on each object.

Let $P_\bullet$ be an object in the category of chains of projective sheaves, $\mathrm{Kom}(\mathcal{P})$. For a chain map, $u\colon P_\bullet \to (P_\bullet)^{!\to *}$, we define

$$\mathrm{Tr}(u) = \sum_i (-1)^i \mathrm{Tr}(u_i),$$

where u_i is the map from P_i to $(P_i)^{!\to *}$.

Theorem 6.8 *The trace of u as above is independent of homotopy class. In particular, it gives rise to trace for each map from an $F \in \mathcal{D}^{\mathrm{b}}(\mathbb{Q}\,(\mathcal{C}))$ to $F^{!\to *}$.*

Proof It suffices to show this on each X-diagonal part, $X \in \mathrm{Ob}\,(\mathcal{C})$. In this case it suffices to show that for a map of finite dimensional vector spaces $d\colon V_1 \to V_0$ and $K\colon V_0 \to V_1$ that $\mathrm{Tr}(Kd) = \mathrm{Tr}(dK)$. But this is a standard fact about traces.

□

Given two morphisms in $\mathcal{D}^{\mathrm{b}}(\mathbb{Q}\,(\mathcal{C}))$, $f\colon P_\bullet \to F_\bullet$ and $g\colon F_\bullet \to P^{!\to *}$, we have $\mathrm{Tr}(gf) \in \mathbb{Q}$; hence we have a map

$$u\colon \mathrm{Hom}(F_\bullet, P^{!\to *}) \to \big(\mathrm{Hom}(P_\bullet, F_\bullet)\big)^*$$

defined for each $F_\bullet$.

Theorem 6.9 *The above map u is natural in both $P_\bullet$ and $F_\bullet$.*

Proof To see naturality in $F_\bullet$, consider a map $g_\bullet\colon F_\bullet \to G_\bullet$; for each $w \in \mathrm{Hom}(G_\bullet, P_\bullet^{!\to *})$ and $v \in \mathrm{Hom}(P_\bullet, F_\bullet)$ we have

$$gv \in \mathrm{Hom}(P_\bullet, G_\bullet), \qquad wg \in \mathrm{Hom}(F_\bullet, P_\bullet^{!\to *}),$$

and naturality amounts to

$$\mathrm{Tr}\big(w(gv)\big) = \mathrm{Tr}\big((wg)v\big),$$

which is clear. Similarly, naturality in $P_\bullet$ reduces to the fact that for each $g\colon P_\bullet \to Q_\bullet$, $w \in \mathrm{Hom}(F_\bullet, P_\bullet^{!\to *})$, and $v \in \mathrm{Hom}(Q_\bullet, F_\bullet)$ we have

$$\mathrm{Tr}\big(w(vg)\big) = \mathrm{Tr}\big((g^{!\to *}w)v\big).$$

This follows by applying μ, whereupon g and $g^{!\to *}$ become identified and the trace identity is standard.

□

Next we show that u gives an isomorphism on objects for the form $(k_A)_!\mathbb{Q}$. Indeed, we have already seen that for $B \in \mathcal{D}^{\mathrm{b}}(\mathbb{Q}(\mathcal{C}))$,

$$\mathrm{Hom}_{\mathcal{D}}((k_A)_!\mathbb{Q}, B) \simeq H^0(k_A^* B),$$

and

$$\mathrm{Hom}_{\mathcal{D}}(B, (k_A)_*\mathbb{Q}) \simeq \bigl(H^0(k_A^* B)\bigr)^*.$$

If $f \in \mathrm{Hom}_{\mathcal{D}}((k_A)_!\mathbb{Q}, B)$ and $f \neq 0$, then restricted to A, f maps 1 to a nonzero element of $H^0(k_A^* B)$. Thus there exists a linear map $\ell\colon H^0(k_A^* B) \to \mathbb{Q}$ such that $\ell(f(1)) \neq 0$. It follows that ℓ gives rise to $g \in \mathrm{Hom}_{\mathcal{D}}(B, (k_A)_*\mathbb{Q})$ such that

$$\mathrm{Tr}(gf) = \ell\bigl(f|_A(1)\bigr) \neq 0.$$

So u for $(k_A)_!\mathbb{Q}$ gives a morphism of vector spaces of the same dimension that has a zero nullspace; thus u is an isomorphism on each $(k_A)_!\mathbb{Q}$.

Since the elements of the form $(k_A)_!\mathbb{Q}$ triangularly generate $\mathcal{D}^{\mathrm{b}}(\mathbb{Q}(\mathcal{C}))$, we conclude the following theorem.

Theorem 6.10 *The above functor u gives an isomophism of functors* $! \to *$ *(followed by vector space Yoneda) with* $\mathrm{L} \to \mathrm{R}$.

7 Local Criterion

We are interested to know for which finite semitopological categories we have $\mathbb{Q}^{!\to *} \simeq \mathbb{Q}[n]$ for an integer n, or $F^{!\to *} \simeq F[n]$ for some F, and similar such conditions. Here we give a necessary (but not sufficient) condition involving a sort of local Euler characteristic.

Let $\mathcal{C}$ be a finite semitopological category with adjacency matrix M, i.e., the matrix indexed on pairs of objects of $\mathcal{C}$ such that $M(X,Y)$ is the size of $\mathrm{Hom}(X,Y)$. For an $F^\bullet \in \mathcal{D}^{\mathrm{b}}(\mathbb{Q}(\mathcal{C}))$ we define v_F to be the vector indexed on $\mathrm{Ob}(\mathcal{C})$ whose X component is

$$v_F(X) = \sum_i (-1)^i \dim(F^i|_X).$$

It is well defined on $\mathcal{D}^{\mathrm{b}}(\mathbb{Q}(\mathcal{C}))$ and we shall call it the local Euler characteristic. Of course, $v_{F[n]} = (-1)^n v_F$ for all n. The following theorem gives a necessary condition for checking duality:

Theorem 7.1 *If $G = F^{!\to *}$, then*

$$v_G = M^{\mathrm{T}} M^{-1} v_F.$$

In particular, if $F^{!\to *} = F[n]$ for some n, then v_F is an eigenvector of $M^{\mathrm{T}} M^{-1}$ with eigenvalue ± 1. Similarly, if $(F^{!\to *})^{!\to *} = F[m]$, then v_F is a linear combination of eigenvectors with eigenvalues ± 1 if m is even, and $\pm i$ if m is odd.

Proof First let $F \simeq P_\bullet$ with $P_i \simeq \bigoplus_X k_{X!} V_{X,i}$. Let w_F be defined by

$$w_F(X) = \sum_i (-1)^i \dim V_{X,i}.$$

Since

$$\dim(k_{Y!}V)(X) = (\dim V)\ \big|\mathrm{Hom}(X,Y)\big|,$$

we have

$$v_F(X) = \sum_i (-1)^i \sum_Y (\dim V_{Y,i})\ \big|\mathrm{Hom}(X,Y)\big| = \sum_Y w_F(Y)\ \big|\mathrm{Hom}(X,Y)\big|.$$

Hence $v_F = M w_F$. Similarly

$$v_G(X) = \sum_i (-1)^i \sum_Y (\dim V_{Y,i})\ \big|\mathrm{Hom}(Y,X)\big|,$$

and we conclude $v_G = M^T w_F$. Thus $v_G = M^T w_F = M^T(M^{-1} v_F)$.

□

8 Strongly n-dimensional morphisms

Definition 8.1 *Let $\phi\colon \mathcal{C} \to \mathcal{S}$ be a morphism of semitopological categories. We say that ϕ is* strongly n-dimensional *if $(! \to *)\phi^*$ is isomorphic to $[n]\phi^*(! \to *)$ as functors from $\mathcal{D}^{\mathrm{b}}(\mathbb{Q}(\mathcal{C}))$ to $\mathcal{D}^{\mathrm{b}}(\mathbb{Q}(\mathcal{S}))$*

The point of this section and the next is to prove that the above notion is stable under base change. We will also give alternative descriptions of strong n-dimensionality. First we note some easy alternative descriptions.

Theorem 8.2 *The following are equivalent:*

1. *ϕ is strongly n-dimensional, i.e.,* $(! \to *)\phi^* \simeq [n]\phi^*(! \to *)$,

2. $\phi^*(* \to !) \simeq [n](* \to !)\phi^*$,

3. $\underline{\underline{L}}\phi_! \simeq [n]\underline{\underline{R}}\phi_*$.

Proof Condition (1) implies (2) by applying $* \to !$ and to left and right sides; similarly (2) implies (1). For condition (3) we have that (1) implies

$$\mathrm{Hom}(F_\bullet, [n]\phi^*(! \to *)P_\bullet) \simeq \mathrm{Hom}(F_\bullet, (! \to *)\phi^* P_\bullet)$$

so

$$\mathrm{Hom}([-n]\underline{\underline{L}}\phi_! F_\bullet, (! \to *)P_\bullet) \simeq \mathrm{Hom}(F_\bullet, (! \to *)\phi^* P_\bullet),$$

so

$$\mathrm{Hom}(P_\bullet, [-n]\underline{\underline{L}}\phi_! F_\bullet) \simeq \mathrm{Hom}(\phi^* P_\bullet, F_\bullet) \simeq H(P_\bullet, \underline{\underline{R}}\phi_* F_\bullet),$$

and vice versa. Yoneda's lemma gives the desired isomophism of functors.

□

Definition 8.3 *We say that $\mathcal{C}$ is strongly n-dimensional if the map from $\mathcal{C}$ to Δ_0 is strongly n-dimensional, or, equivalently, if* $\mathbb{Q}^{! \to *} \simeq \mathbb{Q}[n]$.

We mention that a number of models of n-dimensional manifolds are strongly n-dimensional, not all of them are. For example, let $\mathcal{C}$ be any strongly n-dimensional category, and let $\mathcal{C}'$ be the category obtained by adding one object, X, and one morphism to a minimal element, Y, (and from each morphism from Y adding one corresponding morphism from X). If $\mathcal{C}$ were obtained from a good cover of a manifold, $\mathcal{C}'$ could be obtained by adding one small neighborhood of a point in a minimal open set. It is easy to see that $\mathcal{C}'$ will not be strongly n-dimensional, and indeed, that $(! \to *)\mathbb{Q}$ restricted to X is zero.

On the other hand, we do know a few examples of strongly n-dimensional categories and morphisms. The categories of $(\Delta_1)^n$ with the two extreme objects removed (modelling a good cover of S^{n-2}) is strongly $(n-2)$-dimensional. The category of $n+1$ pairs of objects, modelling a successive upper and lower hemisphere covering of S^n, and its quotient by the cyclic group of order two are strongly n-dimensional. Any Galois morphism or covering morphism (i.e., a morphism that is Galois or a covering space on the underlying graphs, as in [Fri93]) is strongly 0-dimensional (it is immediate that such a morphism is fiberwise 0-dimensional, see below). And (see below) any base change of a strongly n-dimensional morphism is also one. Hence any fiber product of strongly dimensional morphisms is one, and therefore so is any finite projective limit involving only strongly dimensional morphisms (see [SGA4.I.2.3.(iii)]).

We mention that the "Boolean cube" ${\Delta_1}^n$ for $n \geq 1$ is not strongly m-dimensional for any m, and in fact $(! \to *)^3 = [n]\mathrm{Id}$ there.

Question: We mention that at present we know of no $\mathcal{C}$ with a vector bundle $\omega \not\simeq \mathbb{Q}$ such that $\mathbb{Q}^{! \to *} = \omega[n]$ for some n. Can this happen? We also mention that in all examples of strong dimensionality that we know, the "skeleton" (i.e., the objects, X, where $k_{X*}V_X$ with $V_X \neq 0$ is a summand of one of the injectives) of the greedy $\mathbb{Q}$ injective (or projective) resolution is self-dual, in that there is a simple isomorphism from $\mathcal{C}$ to $\mathcal{C}^{\mathrm{opp}}$ that maps the skeleton to itself. Is this necessary?

8.1 Main Result

We now state the main theorem in this section. Consider a base change diagram (i.e., a Cartesian diagram, i.e., where $\mathcal{X}' = \mathcal{X} \times_{\mathcal{S}} \mathcal{S}'$):

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{f'} & \mathcal{S}' \\ g' \downarrow & & \downarrow g \\ \mathcal{X} & \xrightarrow{f} & \mathcal{S} \end{array} \tag{14}$$

As usual, we say that f' *is obtain from* f *via base change with respect to* g.

Theorem 8.4 *Strong n-dimensionality is closed under arbitrary base change; i.e., if f in the above is strongly n-dimensional, then so is f'.*

Our proof of this fact will be to give another characterization of strong n-dimensionality that is easily seen to be closed under arbitrary base change. We call this characterization "fiberwise n-dimensionality," and it characterizes strong n-dimensionality in local terms, in terms of the fiber over objects and the fiber over morphisms.

Definition 8.5 *If $f\colon \mathcal{X} \to \mathcal{S}$ is a functor on finite, semitopological categories, then we say that f is* fiberwise n-dimensional *if for any base change diagram as in equation (14) with $\mathcal{S}' = \Delta_1$ we have that f' is strongly n-dimensional.*

We claim that the notion of fiberwise n-dimensional is easily seen to be stable under base change. Indeed, in a diagram of base changes:

$$\begin{array}{ccccc} \mathcal{X}'' & \xrightarrow{h'} & \mathcal{X}' & \xrightarrow{g'} & \mathcal{X} \\ {\scriptstyle f''}\downarrow & & {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \Delta_1 & \xrightarrow{h} & \mathcal{S} & \xrightarrow{g} & \mathcal{S} \end{array}$$

where f is fiberwise n-dimensional, $g\colon \mathcal{S}' \to \mathcal{S}$ is arbitrary, and $h\colon \Delta_1 \to \mathcal{S}'$ is arbitrary. Then $g \circ h$ maps Δ_1 to $\mathcal{S}$, and so g'' is strongly n-dimensional. Thus f' is fiberwise n-dimensional.

Let us give a corollary of the base change theorem.

Corollary 8.6 *For $i = 1, \ldots, k$, let $f_i\colon \mathcal{C}^i \to \mathcal{S}$ be a strongly n_i-dimensional functor. Then their fiber product is a strongly $n_1 \ldots n_k$-dimensional functor.*

We prove this by induction on k, using the easy fact that if ϕ_1, ϕ_2 are strongly n_1- and n_2- (respectively) dimensional morphisms, then $\phi_2 \circ \phi_1$ is a strongly $n_1 n_2$-dimensional morphism.

9 Proof of the Base Change Theorem

Here we prove the equivalence of strong n-dimensionality and fiberwise n-dimensionality. We also make some further remarks on base change.

9.1 Strong is Stable Under Fully Faithful Base Change

Theorem 9.1 *Let $f\colon \mathcal{X} \to \mathcal{S}$ be a strongly n-dimensional functor of semi-topological categories. Then if $g\colon \mathcal{S}' \to \mathcal{S}$ is an open or closed inclusion, then the base change morphism $f'\colon \mathcal{X}' \to \mathcal{S}'$ is strongly n-dimensional.*

Proof Here is the idea. Let $g\colon \mathcal{S}' \to \mathcal{S}$ be an open inclusion. Consider (1) of Theorem 8.2, i.e,, the definition of $! \to *$; then $! \to *$ on $\mathcal{X}'$ and $\mathcal{S}'$ may be computed on $\mathcal{X}$ and $\mathcal{S}$ (extending sheaves from $\mathcal{X}', \mathcal{S}'$ to sheaves on $\mathcal{X}, \mathcal{S}$ by extension by zero) and restricting back to $\mathcal{X}'$ and $\mathcal{S}'$. Strong n-dimensionality is clearly preserved by this process of extending by zero and then restricting. Thus f' is strongly n-dimensional.

In more detail, we have $g_!$ and $g'_!$ are exact and fully faithful, and the above shows

$$(! \to *)_{\mathcal{X}'} \simeq (g')^*(! \to *)_{\mathcal{X}}\, g'_!, \quad (! \to *)_{\mathcal{S}'} \simeq g^*(! \to *)_{\mathcal{S}}\, g_!$$

(all functors on the appropriate derived category). Furthermore it is easy to see that $g'_!(f')^* \simeq f^* g_!$ since g is an open inclusion. Thus

$$(! \to *)_{\mathcal{X}'}(f')^* \simeq (g')^*(! \to *)_{\mathcal{X}}\, g'_!(f')^* \simeq (g')^*(! \to *)_{\mathcal{X}}\, f^* g_!$$

$$\simeq (g')^*[n]f^*(! \to *)_{\mathcal{S}}\, g_! \simeq [n](f')^* g^*(! \to *)_{\mathcal{S}}\, g_! \simeq [n](f')^*(! \to *)_{\mathcal{S}'}.$$

Similarly condition (2) of Theorem 8.2 shows that strong dimensionality is invariant under closed inclusion base change.

□

Along similar but more involved lines we shall prove that strong n-dimensionality is stable under fully faithful base change.

Theorem 9.2 *Let $f\colon \mathcal{X} \to \mathcal{S}$ be a strongly n-dimensional functor of semi-topological categories. Then if $g\colon \mathcal{S}' \to \mathcal{S}$ is fully faithful, then the base change morphism $f'\colon \mathcal{X}' \to \mathcal{S}'$ is strongly n-dimensional.*

Lemma 9.3 *Let $f\colon \mathcal{X} \to \mathcal{S}$ be strongly n-dimensional. For $S \in \mathrm{Ob}(\mathcal{S})$, let $\mathcal{X}_S$ denote the full subcategory of $\mathcal{X}$ with objects $f^{-1}(S)$. Then a projective resolution of $\mathbb{Q}$ on $\mathcal{X}_S$ extends naturally to one of $f^*k_{S!}\mathbb{Q}$ on $\mathcal{X}$ in the following sense: let $\cdots \to P_1 \to P_0 \to \mathbb{Q}$ be a projective resolution of $\mathbb{Q}$ on $\mathcal{X}_S$ with P_i a sum of $k_{X!}V_{X,i}$ over $X \in f^{-1}(S)$; then the sums of $k_{X!}V_{X,i}$ on $\mathcal{X}$ (i.e., where now $k_X\colon \Delta_0 \to \mathcal{X}$ rather than $\Delta_0 \to \mathcal{X}_S$ as before) give a projective resolution, $\tilde{P}_i$, of $f^*k_{S!}\mathbb{Q}$. We have the analogous statement about an injective resolution of $\mathbb{Q}$ on $\mathcal{X}_S$ extending to one of $f^*k_{S*}\mathbb{Q}$ on $\mathcal{X}$. Finally, if $\tilde{I}^{n-i}$ is the sum of $k_{X*}V_{X,i}$ on $\mathcal{X}$ (with $V_{X,i}$ as above), then we have an injective resolution of $f^*k_{S*}\mathbb{Q} \to \tilde{I}^0 \to \tilde{I}^1 \to \cdots$*

Proof Consider a projective resolution of $\mathbb{Q}$ on $\mathcal{X}_S$ as in the statement of the theorem. By Theorem 9.1 we have f restricted to $\mathcal{X}_S$ is strongly n-dimensional, and hence $I^j = \oplus_X k_{X*}V_{X,n-j}$ gives an injective resolution of $\mathbb{Q}$ restricted to $\mathcal{X}_S$. We apply strong dimensionality, in the form $f^*(* \to !) \simeq [n](* \to !)f^*$ to $k_{S*}\mathbb{Q}$ to conclude

$$f^*k_{S!}\mathbb{Q} \simeq [n](* \to !)f^*k_{S*}\mathbb{Q}.$$

To find a projective resolution of $f^* k_{S!}\mathbb{Q}$, it suffices to do so when $\mathcal{S}$ is replaced by the smallest open set containing S (and $\mathcal{X}$ replaced by f^{-1} of this open set). So we may assume that S is a terminal object of $\mathcal{S}$. In this case $f^* k_{S*}\mathbb{Q}$ is just $\mathbb{Q}_{f^{-1}(S)}$ (i.e., $\mathbb{Q}$ on the closed set $f^{-1}(S)$ and zero elsewhere), in which case $\tilde{I}^j = \oplus_X k_{X*} V_{X,n-j}$ viewed on $\mathcal{X}$ is visibly an injective resolution (each $\tilde{I}^j$ is zero outside the closed set $f^{-1}(S)$ and restricts to I^j on $f^{-1}(S)$). We therefore have

$$f^* k_{S!}\mathbb{Q} \simeq [n](* \to !)\tilde{I}^\bullet.$$

But $(* \to !)\tilde{I}^j \simeq k_{X!} V_{X,n-j}$, so $[n](* \to !)\tilde{I}^\bullet$ is isomorphic to the projective resolution $\tilde{P}_\bullet$. Thus

$$f^* k_{S!}\mathbb{Q} \simeq \tilde{P}_\bullet.$$

The statement regarding extending injective resolutions works by reversing arrows. The last statement holds by using the extension of an injective resolution, given that the I^j as above give an injective resolution of $\mathbb{Q}$ restricted to $\mathcal{S}$.

□

Lemma 9.4 *Let f be strongly n-dimensional and g be fully faithful in the cartesian diagram of equation (14). Then there are isomorphisms $f^* \underline{\underline{R}} g_* \to (\underline{\underline{R}} g'_*) f'^*$ and $(\underline{\underline{L}} g'_!) f'^* \to f^* \underline{\underline{L}} g_!$.*

Proof We claim that for any $Q \in \mathrm{Ob}(\mathcal{S}')$ we have

$$f^* \underline{\underline{R}} g_* k_{Q*}\mathbb{Q} \simeq (\underline{\underline{R}} g'_*) f'^* k_{Q*}\mathbb{Q}.$$

Indeed, $\underline{\underline{R}} g_* k_{Q*}\mathbb{Q} \simeq k_{g(Q)*}\mathbb{Q}$, so the left-hand-side of the above displayed equation is simply $f^* k_{g(Q)*}\mathbb{Q}$. But an injective resolution of this sheaf, by the previous lemma, is given by the extension of a resolution of $\mathbb{Q}$ on $\mathcal{X}_{g(Q)}$; since g is fully faithful so is g' and hence g' restricts to an isomorphism $\mathcal{X}'_Q \to \mathcal{X}_{g(Q)}$, which is to say it is isomorphic (by the previous lemma) to the image under $\underline{\underline{R}} g'_*$ to the injective resolution extended from that of $\mathbb{Q}$ on $\mathcal{X}'_Q$, which is the right-hand-side of the above equation.

Next we claim that the adjunctive map $\mathrm{Id} \to (\underline{\underline{R}} g'_*) g'^*$ applied to any H in the image of $\underline{\underline{R}} g'_*$ is an isomorphism. Indeed, let $\mathcal{H}$ be subcategory of $\mathcal{D}^{\mathrm{b}}(\mathbb{Q}(\mathcal{X}))$ triangularly generated by $k_{S*}\mathbb{Q}$ over S in the image of g'. Since g' is fully faithful, we see that $g' \colon \mathcal{H} \to \mathcal{D}^{\mathrm{b}}(\mathbb{Q}(\mathcal{X}'))$ is fully faithful. Furthermore $(\underline{\underline{R}} g'_*) k_{T*}\mathbb{Q}$, for each $T \in \mathrm{Ob}(\mathcal{X}')$, is isomorphic to $k_{g'(T)*}\mathbb{Q}$; thus the image of $\underline{\underline{R}}$ lies in $\mathcal{H}$. It follows by the end of Section 2.1 that $\mathrm{Id} \to (\underline{\underline{R}}g'_*)g'^*$ is an isomorphism on the objects of $\mathcal{H}$.

We know that $f^*\underline{\underline{R}}g_*k_{Q*}\mathbb{Q}$ is in the image of $\underline{\underline{R}}g'_*$ for all $Q \in \mathrm{Ob}\,(\mathcal{S}')$. Thus the morphism obtained from the adjunctive morphism,

$$f^*\underline{\underline{R}}g_* \to (\underline{\underline{R}}g'_*)g'^*f^*\underline{\underline{R}}g_*$$

is an isomorphism on all $k_{Q*}\mathbb{Q}$, and hence everywhere (by closing triangularly). Since g is fully faithful and $fg' = gf'$, we have

$$g'^*f^*\underline{\underline{R}}g_* = f'^*g^*\underline{\underline{R}}g_* = f'^*,$$

giving the isomorphism $f^*\underline{\underline{R}}g_* \simeq (\underline{\underline{R}}g'_*)f'^*$.

The other isomorphism comes from reversing the arrows.

□

Lemma 9.5 *Let $g\colon \mathcal{S}' \to \mathcal{S}$ be an arbitrary map. Then*

$$\underline{\underline{R}}g_*(! \to *)_{\mathcal{S}'} \simeq (! \to *)_{\mathcal{S}}\underline{\underline{L}}g_! \tag{15}$$

Furthermore, in any diagram as in equation (14), with f strongly n-dimensional, there is a canonical map

$$\mu\colon \underline{\underline{R}}g'_*[-n](! \to *)_{\mathcal{X}'}f'* \to (\underline{\underline{R}}g'_*)f'^*(! \to *)_{\mathcal{S}'} \tag{16}$$

once we give morphisms $(\underline{\underline{L}}g'_!)f'^* \to f^*\underline{\underline{L}}g_!$ and $f^*\underline{\underline{R}}g_* \to (\underline{\underline{R}}g'_*)f'^*$; if the two given morphisms are isomorphisms, then so is μ above.

Proof For all $F \in \mathcal{D}^{\mathrm{b}}(\mathbb{Q}(\mathcal{S}))$ and $G \in \mathcal{D}^{\mathrm{b}}(\mathbb{Q}(\mathcal{S}'))$ we have

$$\begin{aligned}\mathrm{Hom}(F, (! \to *)(\underline{\underline{L}}g_!)G) &\simeq \mathrm{Hom}((\underline{\underline{L}}g_!)G, F) \simeq \mathrm{Hom}(G, g^*F)\\ &\simeq \mathrm{Hom}(g^*F, (! \to *)G) \simeq \mathrm{Hom}(F, (\underline{\underline{R}}g_*)(! \to *)G)\end{aligned}$$

with the isomorphisms functorial in F, G. This proves equation (15).

For the second part have morphisms

$$\begin{aligned}\underline{\underline{R}}g'_*[-n](! \to *)f'^* &\to [-n](! \to *)(\underline{\underline{L}}g'_!)f'^* \to [-n](! \to *)f^*\underline{\underline{L}}g_!\\ &\to f^*(! \to *)\underline{\underline{L}}g_! \to f^*\underline{\underline{R}}g_*(! \to *) \to (\underline{\underline{R}}g'_*)f'^*(! \to *).\end{aligned}$$

□

Now we finish proving the theorem about stability of strong dimensionality under fully faithful base change, by simply multiplying equation 16 by g'^* on the left and using that $g'^*\underline{\underline{R}}g'_* \simeq \mathrm{Id}$.

□

9.2 Strong is Stable Under Special Base Chage

For integer $m \geq -1$, let L_m be the category with two objects $0,1$ and with $m+1$ morphisms from 0 to 1 (and no other nonidentity morphisms); we call L_m the *bouquet of m loops*. Of course, $\Delta_1 \simeq L_0$.

Definition 9.6 *By a* special functor*, we mean a functor $u\colon \Delta_1 \to L_m$ for some $m \geq 1$ such that the objects $0,1$ in Δ_1 are mapped to the same in L_m.*

Let $\mathcal{S}$ be an arbitrary category such that $\mathrm{Hom}_{\mathcal{S}}(A,B)$ is a finite set for all $A,B \in \mathrm{Ob}\,(\mathcal{S})$. Any functor $\Delta_1 \to \mathcal{S}$ factors essentially uniquely as a special morphism followed by a fully faithful morphism. In this subsection we show that strong n-dimensionality is stable under special base change, and therefore any base change from Δ_1.

Definition 9.7 *We say that a functor $f\colon \mathcal{X} \to \mathcal{S}$ has the* target lifting property *if for each $\phi \in \mathrm{Fl}\,(\mathcal{S})$ and for each $T \in \mathrm{Ob}\,(\mathcal{X})$ such that $f(T)$ is the target of ϕ, there is a $\xi \in \mathrm{Fl}\,(\mathcal{X})$ whose target is T and such that $f(\xi) = \phi$. The* source lifting property *is similarly defined.*

Theorem 9.8 *Let $f\colon \mathcal{X} \to \mathcal{S}$ be strongly n-dimensional. Then f has the source and target lifting property.*

Proof Assume that $f\colon \mathcal{X} \to \mathcal{S}$ is strongly n-dimensional. Let $S,T \in \mathrm{Ob}\,(\mathcal{S})$, and let

$$\mathrm{Hom}_{\mathcal{S}}(S,T) = \{\phi = \phi_1, \phi_2, \ldots, \phi_m\}.$$

After an open inclusion and a closed inclusion we may assume that S is a minimal object of $\mathcal{S}$ and T is a maximal one. Let $\mathbb{Q}|_{f^{-1}(S)}$ have projective resolution of the form $P_\bullet$ with $P_i = \bigoplus_{f(X)=S} (k_X)_! V_{X,i}$. Then $\mathbb{Q}_{f^{-1}(S)}$ (the sheaf $\mathbb{Q}$ on $f^{-1}(S)$ extended by zero to the rest of $\mathcal{X}$) has the same projective resolution as $\mathbb{Q}|_{f^{-1}(S)}$, provided the $(k_X)_! V_{X,i}$ are viewed on $\mathbb{Q}(\mathcal{X})$ (as opposed to just $\mathbb{Q}(f^{-1}(S))$). Hence

$$(! \to *) f^* \mathbb{Q}_S = (! \to *) \mathbb{Q}_{f^{-1}(S)} = I^\bullet,$$

where

$$I^{-i} = \bigoplus_{f(X)=S} (k_X)_* V_{X,i}.$$

On the other hand set

$$F = f^*(! \to *)\mathbb{Q}_S = f^*((k_S)_*\mathbb{Q});$$

for $Y \in \mathrm{Ob}\,(\mathcal{X})$ and $\xi\colon X \to Y$ with $f(X) = S$ we have

$$F(Y) = \mathbb{Q}^{\mathrm{Hom}(S,f(Y))},$$

and $F\xi$ is zero on all components of $F(Y)$ except the one corresponding to $f(\xi)$, on which it is the identity map to $F(X) = \mathbb{Q}$.

First we claim that f has the target lifting property. Indeed, consider the functor $K\colon \mathbb{Q}(\mathcal{X}) \to \mathbb{Q}(f^{-1}(T))$ given by

$$(KG)(Y) = \bigcap_{i\geq 2,\, f(\xi)=\phi_i,\, \mathrm{targ}(\xi)=Y} \ker G\xi; \tag{17}$$

K is left exact, and so we have $\underline{\underline{R}}K\colon \mathcal{D}^{\mathrm{b}}(\mathbb{Q}(\mathcal{X})) \to \mathcal{D}^{\mathrm{b}}(\mathbb{Q}(f^{-1}(T)))$. Fix $Y \in \mathrm{Ob}\,(\mathcal{X})$ with $f(Y) = T$. On the one hand, it is easy to see that if there is no ξ with target Y and with $f(\xi) = \phi_1$, then for any X with $f(X) = S$ we have $(K(k_X)_*V_{X,i})(Y)$ vanishes. On the other hand, we have KF includes a copy of $\mathbb{Q}$ (corresponding to ϕ_1), and hence $KF \not\simeq 0$. It follows that $(\underline{\underline{R}}K)F \not\simeq 0$ (since $H^0((\underline{\underline{R}}K)F) = KF$ since F is a sheaf), and hence $(\underline{\underline{R}}K)I^\bullet[-n] \not\simeq 0$. Thus K cannot vanish at Y when applied all the components $(k_X)_*V_{X,i}$ of I^i for all i (since $(\underline{\underline{R}}K)I^\bullet$ can be computed by applying K to each I^i, since each I^i is injective). Therefore f has the target lifting property.

By symmetry, f has the source lifting property.

□

We mention that a strongly n-dimensional f need not be prefibered (see [SGA1.VI.6.1]). Indeed, let $\mathcal{C}, \mathcal{C}'$ respectively be copies of L_1, with objects $\{a, b\}$ and $\{a', b'\}$ respectively. Consider the union of $\mathcal{C}$ and $\mathcal{C}'$ with four additional arrows: one from a or b to a' or b'. We call these four addition arrows "a collection of zero arrows," since they do not affect the projective or injective resolutions of $\mathbb{Q}, \mathbb{Q}_{\mathcal{C}}, \mathbb{Q}_{\mathcal{C}'}$. Now take two copies of $\mathcal{C}$ lying over a point, extend the base by Δ_1, to get a strongly 1-dimensional $f'\colon \mathcal{C}'' \to \Delta_1$; $\mathcal{C}''$ has two connected components. Now take one component of $f'^{-1}(0)$ and connect it to one of $f'^{-1}(1)$ via "zero arrows" as above. The resulting morphism, f'', is still strongly 1-dimensional, but is not prefibered, as there is no inverse image by $0 \to 1$ of a point, P, in $f'^{-1}(1)$ in the component of the zero arrows (in particular, the category whose objects are the morphisms over $0 \to 1$ with target P has no initial element).

Let us return to a base change map as in equation (14) with $\mathcal{S}' = \Delta_1$, $\mathcal{S} = L_m$, $g(0) = 0$, $g(1) = 1$, $g(0 \to 1) = \phi_1$, and $\mathrm{Hom}_{\mathcal{S}}(0, 1) = \{\phi_1, \ldots, \phi_m\}$.

We take the functor K of equation (17) and extend it to all of $\mathbb{Q}(\mathcal{X}')$ by setting $K_{\mathrm{big}} \colon \mathbb{Q}(\mathcal{X}) \to \mathbb{Q}(\mathcal{X}')$ via

$$(K_{\mathrm{big}}F)(X) = \begin{cases} F(g'(X)) & \text{if } f'(X) = 0, \\ (KF)(X) & \text{if } f'(X) = 1. \end{cases}$$

We check that K_{big} extends to a functor (i.e., naturally acts on $\mathbb{Q}(\mathcal{X})$ morphisms). Define $Z_{\mathrm{big}} \colon \mathbb{Q}(\mathcal{X}') \to \mathbb{Q}(\mathcal{X})$ via $Z_{\mathrm{big}}F$ is F on $g'(\mathcal{X}')$ and $(Z_{\mathrm{big}}F)\phi_i = 0$ for $i \geq 2$. Define ${K_{\mathrm{big}}}' \colon \mathbb{Q}(\mathcal{X}) \to \mathbb{Q}(\mathcal{X}')$ via

$$({K_{\mathrm{big}}}'F)(X) = \begin{cases} K'(X) & \text{if } f'(X) = 0, \\ F(g'(X)) & \text{if } f'(X) = 1, \end{cases}$$

where

$$(K'F)(X) = F(g'(X)) / \sum_{\xi \text{ s.t. } f(\xi) \neq \phi_1, s\xi = g'(X)} \mathrm{image} F(\xi).$$

We define $K_{\mathrm{small}}, Z_{\mathrm{small}}, {K_{\mathrm{small}}}'$ to be $K_{\mathrm{big}}, Z_{\mathrm{big}}, {K_{\mathrm{big}}}'$ (respectively) in the case where $f = \mathrm{Id}$; in other words, $K_{\mathrm{small}}, {K_{\mathrm{small}}}'$ are maps $\mathbb{Q}(\mathcal{S}') \to \mathbb{Q}(\mathcal{S})$

$$(K_{\mathrm{small}}F)(X) = \begin{cases} F(0) & \text{if } X = 0, \\ \bigcap_{i \geq 2} \ker(F\phi_i) & \text{if } X = 1, \end{cases}$$

$$({K_{\mathrm{small}}}'F)(X) = \begin{cases} F(0) / \sum_{i \geq 2} \mathrm{image}(\phi_i) & \text{if } X = 0, \\ F(1) & \text{if } X = 1, \end{cases}$$

and $Z_{\mathrm{small}} \colon \mathbb{Q}(\mathcal{S}) \to \mathbb{Q}(\mathcal{S}')$ is given by $(Z_{\mathrm{small}}F)(X) = F(X)$ for $X = 0, 1$, $(Z_{\mathrm{small}}F)(\phi) = F(0 \to 1)$, and $(Z_{\mathrm{small}}F)(\phi_i) = 0$ for $i \geq 2$.

Theorem 9.9 *We have ${K_{\mathrm{small}}}', Z_{\mathrm{small}}, K_{\mathrm{small}}$ and ${K_{\mathrm{big}}}', Z_{\mathrm{big}}, K_{\mathrm{big}}$ are sequences of adjoints. In particular $\underline{\underline{L}}{K_{\mathrm{small}}}' = (* \to !)\underline{\underline{R}}K_{\mathrm{small}}(! \to *)$ and similarly for ${K_{\mathrm{big}}}'$ and K_{big}. Also $K_{\mathrm{big}}f^* = f'^* K_{\mathrm{small}}$ and ${K_{\mathrm{big}}}'f^* = f'^* {K_{\mathrm{small}}}'$. Finally, $K_{\mathrm{big}}g'_* \simeq \mathrm{Id}$, $(\underline{\underline{R}}K_{\mathrm{big}})(\underline{\underline{R}}g'_*) \simeq \mathrm{Id}$, and similarly for ${K_{\mathrm{big}}}'$ and $g'_!$ (and $\underline{\underline{L}}$ replacing $\underline{\underline{R}}$), and similarly for K_{small}, g replacing K_{big}, g' (respectively).*

Proof The first sentence (about adjointness) is a simple calculation we mostly leave to the reader; as an example, if $\mu \in \mathrm{Hom}(Z_{\mathrm{big}}F, G)$ and $v \in F(X)$ with $X \in f^{-1}(1)$, then since $(Z_{\mathrm{big}}F)\xi = 0$ for all ξ with $f(\xi) \neq \phi_1$, we have $\mu(X)v \in \ker(G\xi)$ for all ξ with $f(\xi) \neq \phi$, and as such μ gives rise to an element of $\mathrm{Hom}(F, K_{\mathrm{big}}G)$; similarly there is an inverse map from $\mathrm{Hom}(F, K_{\mathrm{big}}G)$ to $\mathrm{Hom}(Z_{\mathrm{big}}F, G)$, and similarly a bijection $\mathrm{Hom}({K_{\mathrm{big}}}'F, G) \to \mathrm{Hom}(F, Z_{\mathrm{big}}G)$ (of course, the case ${K_{\mathrm{small}}}', Z_{\mathrm{small}}, K_{\mathrm{small}}$ is a special case of ${K_{\mathrm{big}}}', Z_{\mathrm{big}}, K_{\mathrm{big}}$).

The second sentence follows from equation (7) using the adjointness of the first sentence.

The third sentence follows almost immediately from the fact that f has the target lifting property (for the first equation) and source lifting (for the second).

In the fourth sentence, the case with K_{small}, g replacing K_{big}, g' is just a special case. So we need only deal with the K_{big}, g' case. That $K_{\mathrm{big}} g'_* = \mathrm{Id}$ follows from the target lifting property of f. Since g'_* takes injectives to injectives (since it has an exact left adjoint), we have $\underline{\underline{R}}(K_{\mathrm{big}})\underline{\underline{R}}g'_* = \underline{\underline{R}}(K_{\mathrm{big}} g'_*) = \mathrm{Id}$. Similarly for ${K_{\mathrm{big}}}' g'_! = \mathrm{Id}$ and $\underline{\underline{L}}({K_{\mathrm{big}}}')\underline{\underline{L}}g'_! = \mathrm{Id}$.

□

Theorem 9.10 *We have* $\underline{\underline{R}}(K_{\mathrm{big}} f^*) = (\underline{\underline{R}}K_{\mathrm{big}}) f^*$.

Proof It suffices to see that if $Y = 0$ or $Y = 1$, then K_{big} is exact on the injective resolution of $f^* k_{Y*}\mathbb{Q}$ (in other words, $f^* k_{Y*}\mathbb{Q}$ is K_{big}-acyclic). But the injective resolution of $f^* k_{Y*}\mathbb{Q}$ consists of components of the form $k_{X*}\mathbb{Q}$ for X's with $f(X) = Y$, using Lemma 9.3. Furthermore, $K_{\mathrm{big}} k_{X*}\mathbb{Q} \simeq k_{\tilde{X}*}\mathbb{Q}$ for the unique $\tilde{X}$ with $g'(\tilde{X}) = X$, and K_{big} becomes an equivalence of categories when restricted to the category triangularly generated by the $k_{X*}\mathbb{Q}$ for X with $f(X) = Y$; this follows from the fact that if $f(x_1) = f(x_2)$ then

$$\mathrm{Hom}_{\mathbb{Q}(\mathcal{X}')}(k_{\tilde{X}_1 *}\mathbb{Q}, k_{\tilde{X}_2 *}\mathbb{Q}) \simeq \big(\mathrm{Hom}(\mathbb{Q}, \mathbb{Q})\big)^{\mathrm{Hom}_{\mathcal{X}'}(\tilde{X}_1, \tilde{X}_2)}$$

$$\simeq \big(\mathrm{Hom}(\mathbb{Q}, \mathbb{Q})\big)^{\mathrm{Hom}_{\mathcal{X}}(X_1, X_2)} \simeq \mathrm{Hom}_{\mathbb{Q}(\mathcal{X})}(k_{X_1 *}\mathbb{Q}, k_{X_2 *}\mathbb{Q})$$

for $X_i = g'(\tilde{X}_i)$. Hence the desired exactness (or acyclicity) of K_{big}.

□

We remark that while K_{big} is an equivalence of categories when restricted to the subcategory triangularly generated by $k_{X*}\mathbb{Q}$ over X with either $f(X)=0$ or $f(X)=1$, K_{big} is not generally right exact. Here is a simple example. Consider on $\mathbb{Q}(L_1)$ the surjection $F\to G$ where $F=k_{0*}\mathbb{Q}$, $G=k_{1*}\mathbb{Q}$, and where $F\phi_1(a,b)=a$, $F\phi_2(a,b)=b$, and $F(1)\to G(1)$ is given by $(a,b)\mapsto b$. Then $F\to G$ is indeed a surjection, but $K_{\mathrm{big}}F\to K_{\mathrm{big}}G$ is the zero map, not a surjection.

We now finish the special base change theorem. We have

$$\underline{\underline{R}}(K_{\mathrm{big}})f^* \simeq \underline{\underline{R}}(K_{\mathrm{big}}f^*) \simeq \underline{\underline{R}}(f'^*K_{\mathrm{small}}) \simeq f'^*(\underline{\underline{R}}K_{\mathrm{small}}),$$

the last equality since K_{small} clearly maps injectives to injectives ($K_{\mathrm{small}}k_{X*}\mathbb{Q}\simeq k_{X*}\mathbb{Q}$ with $X=0$ or $X=1$, the first k_{X*} interpreted in $\mathbb{Q}(\mathcal{S})$, the second in $\mathbb{Q}(\mathcal{S}')$). It follows that

$$f'^* = \underline{\underline{R}}(K_{\mathrm{big}})f^*\underline{\underline{R}}g_*.$$

In the same way we conclude $f'^*=\underline{\underline{L}}(K_{\mathrm{big}}{}')f^*\underline{\underline{L}}g_!$. Hence

$$(!\to *)f'^* \simeq (!\to *)\underline{\underline{R}}(K_{\mathrm{big}})f^*\underline{\underline{R}}g_* \simeq \underline{\underline{L}}(K_{\mathrm{big}}{}')(!\to *)f^*\underline{\underline{R}}g_*$$

$$\simeq \underline{\underline{L}}(K_{\mathrm{big}}{}')[n]f^*(!\to *)\underline{\underline{R}}g_* \simeq \underline{\underline{L}}(K_{\mathrm{big}}{}')[n]f^*(\underline{\underline{L}}g_!)(!\to *) \simeq [n]f'^*(!\to *).$$

9.3 Fiberwise implies strong

Let $f\colon \mathcal{X}\to\mathcal{S}$ be a fiberwise n-dimensional functor between semitopological categories, and let

$$\cdots \to \bigoplus_{X\in f^{-1}(S)} k_{X!}V_{X,1} \to \bigoplus_{X\in f^{-1}(S)} k_{X!}V_{X,0} \to f^*k_{S!}\mathbb{Q}$$

be a fixed projective resolution of of $f^*k_{S!}\mathbb{Q}$ for each $S\in\mathrm{Ob}\,(\mathcal{S})$ (which exists by Lemma 9.3). We may (and shall) assume that $(!\to *)f^*k_{S!}$ is the above projective resolution with $k_{X!}$ replaced by k_{X*}; we also assume that $(!\to *)_{\mathcal{S}}k_{S!}V=k_{S*}V$ for all $S\in\mathrm{Ob}\,(\mathcal{S})$.

We wish to exhibit an isomorphism $\mu\colon \nu_1\to\nu_2$, where $\nu_1=(!\to *)f^*$ and $\nu_2=[n]f^*(!\to *)$. Here is the overall strategy. Say that a $Q_\bullet\in\mathcal{D}^{\mathrm{b}}(\mathbb{Q}(\mathcal{S}))$ is *simple* if

$$Q_i = \bigoplus_{X\in\mathrm{Ob}(\mathcal{S})} (k_X)_!W_{X,i} \tag{18}$$

for some vector spaces $V_{X,i}$. First we will define μ on simple objects. Second, we show that μ defines a natural transformation on the full subcategory of whose objects are the simple ones. Third, we extend μ by general principles, using the fact that any element of the derived category is isomorphic to a simple one, i.e., simple objects are representative.

So fix a simple Q as above and as in equation (18). Consider the diagram below:

$$\begin{array}{ccccc} Q_\bullet = (\oplus k_{S!} W_{S,i})_{i\in\mathbb{Z}} & \xrightarrow{*\to !} & (\oplus k_{S*} W_{S,i})_{i\in\mathbb{Z}} & \xrightarrow{[n]f^*} & (\oplus f^* k_{S*} W_{S,i+n})_{i\in\mathbb{Z}} \\ \downarrow f^* & & & & \uparrow \mu \\ (\oplus f^* k_{S!} W_{S,i})_{i\in\mathbb{Z}} & \xrightarrow{\simeq} & (\oplus k_{X!} V_{X,j} \otimes W_{S,i})_{i,j\in\mathbb{Z}} & \xrightarrow{!\to *} & (\oplus k_{X*} V_{X,j} \otimes W_{S,i})_{i,j\in\mathbb{Z}} \end{array}$$

The arrow labelled $\simeq$ arises since for any $S \in \mathrm{Ob}\,(\mathcal{S})$ we have that $f^* k_{S!}\mathbb{Q}$ has projective resolution

$$\left(\bigoplus_{X\in f^{-1}(S)} k_{X!} V_{X,i} \right)_{i\in\mathbb{Z}} .$$

But note that the composition of $\simeq$ with $! \to *$ in the diagram above is just $! \to *$. The two double complexes in the diagram can be considered a single complex (and therefore elements of the derived category) by the usual diagonal collapse. We define μ via Lemma 9.3 and the isomorphism between

$$\cdots \to \bigoplus_{X\in f^{-1}(S)} k_{X*} V_{X,1} \to \bigoplus_{X\in f^{-1}(S)} k_{X*} V_{X,0} \to 0 \quad \text{and} \quad f^* k_{S*}\mathbb{Q}.$$

Let $\phi\colon Q \to Q'$ be a map of simple objects. We wish to verify the commutativity of the diagram

$$\begin{array}{ccc} (! \to *) f^* Q_1 & \xrightarrow{(!\to *)f^*\phi} & (! \to *) f^* Q' \\ \mu Q_1 \downarrow & & \downarrow \mu Q' \\ [n] f^* (! \to *) Q_1 & \xrightarrow{[n]f^*(!\to *)\phi} & [n] f^* (! \to *) Q' \end{array} \tag{19}$$

Let $S, T \in \mathrm{Ob}\,(\mathcal{S})$ and $\psi \in \mathrm{Hom}(S, T)$, and consider the special case $Q = k_{S!}\mathbb{Q}$, $Q' = k_{T!}\mathbb{Q}$; since

$$\mathrm{Hom}(Q, Q') \simeq \big(\mathrm{Hom}(\mathbb{Q}, \mathbb{Q})\big)^{\mathrm{Hom}(S,T)}$$

(with a canonical isomorphism) we set ϕ to $1^!_\psi$ which we define to be the element of $\mathrm{Hom}(Q,Q')$ that is (according to the right-hand-side of the equation displayed above) the identity on ψ and zero elsewhere. Let's first look at

$$(! \to *)f^*1^!_\psi.$$

We have $f^*1^!_\psi\colon f^*k_{S!}\mathbb{Q} \to f^*k_{T!}\mathbb{Q}$, and we get maps unique up to homotopy

$$\begin{array}{ccccccc}
\cdots \xrightarrow{d_2} & \bigoplus_{X\in f^{-1}(S)} k_{X!}V_{X,1} & \xrightarrow{d_1} & \bigoplus_{X\in f^{-1}(S)} k_{X!}V_{X,0} & \xrightarrow{d_0} & f^*k_{S!}\mathbb{Q} \\
& \downarrow{\scriptstyle \gamma_1} & & \downarrow{\scriptstyle \gamma_0} & & \downarrow{\scriptstyle f^*1^!_\psi} \\
\cdots \xrightarrow{d'_2} & \bigoplus_{Y\in f^{-1}(T)} k_{Y!}V_{Y,1} & \xrightarrow{d'_1} & \bigoplus_{Y\in f^{-1}(T)} k_{Y!}V_{Y,0} & \xrightarrow{d'_0} & f^*k_{T!}\mathbb{Q}
\end{array} \tag{20}$$

We claim that we may assume all the vertical arrows γ_i "involve morphisms only over ψ"; let us make this precise. Recall that we construct γ_0, then γ_1, etc. using projectivity; assume that we have chosen γ_0 via projectivity, so that $d'_0\gamma_0 = (f^*1^!_\psi)d_0$. We have $f^*1^!_\psi$ composed with d_0 determines an element of

$$\mathrm{Hom}(k_{X!}V_{X,0}, f^*k_{T!}\mathbb{Q}) \simeq \big(\mathrm{Hom}(V_{X,0},\mathbb{Q})\big)^{\mathrm{Hom}(f(X),T)}$$

for each $X \in f^{-1}(S)$ (of course, $f(X) = S$). The application of $f^*1^!_\psi$ here means this element is in the image of ψ from

$$\big(\mathrm{Hom}(V_{X,0},\mathbb{Q})\big)^{\mathrm{Hom}(f(X),S)}.$$

Since $\mathrm{Hom}(f(X),S) = \mathrm{Hom}(S,S) = \{\mathrm{Id}_S\}$, we have that

$$(f^*1^!_\psi)\circ d_0 \in \big(\mathrm{Hom}(V_{X,0},\mathbb{Q})\big)^{\mathrm{Hom}(f(X),T)}$$

is zero on all $\mathrm{Hom}(f(X),T) = \mathrm{Hom}(S,T)$ components except possibly $\psi \in \mathrm{Hom}(S,T)$. This means that for any $Y \in f^{-1}(T)$, we have can assume that γ_0 restricts to a map in

$$\mathrm{Hom}(k_{X!}V_{X,0}, k_{Y!}V_{Y,0}) \simeq \big(\mathrm{Hom}(V_{X,0},V_{Y,0})\big)^{\mathrm{Hom}(X,Y)}$$

that is zero on all $\mathrm{Hom}(X,Y)$ components not in $f^{-1}(\psi)$ (by setting γ_0 to 0 there and keeping γ_0 unchanged on its $f^{-1}(\psi)$ components) and obtain a commuting square $(f^*1^!_\psi)\circ d_0 = d'_0\circ\gamma_0$. Then $\gamma_0\circ d_1$ vanishes on $\mathrm{Hom}(X,Y)$

morphisms not in $f^{-1}(\psi)$, and we may similarly assume γ_1 (chosen by projectivity to satisfy $\gamma_0 d_1 = d_0\gamma_1$) has the same property. Similarly for all γ_i's.

The upshot is that equation (20) can be restricted to $\mathcal{X}'$ via the base change $g\colon \mathcal{S}' = \Delta_1 \to \mathcal{S}$ with $g(0 \to 1) = \psi$ without loss of information. So $(! \to *)f^*1^!_\psi$ is (without loss of generality) the map

$$\begin{array}{ccccccc} \cdots & \xrightarrow{(!\to *)d_2} & \bigoplus_{X\in f^{-1}(S)} k_{X*}V_{X,1} & \xrightarrow{(!\to *)d_1} & \bigoplus_{X\in f^{-1}(S)} k_{X*}V_{X,0} & \longrightarrow & 0 \\ & & \Big\downarrow {\scriptstyle (!\to *)\gamma_1} & & \Big\downarrow {\scriptstyle (!\to *)\gamma_0} & & \Big\downarrow \\ \cdots & \xrightarrow{(!\to *)d_2'} & \bigoplus_{Y\in f^{-1}(T)} k_{Y*}V_{Y,1} & \xrightarrow{(!\to *)d_1'} & \bigoplus_{Y\in f^{-1}(T)} k_{Y*}V_{Y,0} & \longrightarrow & 0 \end{array} \qquad (21)$$

where in this diagram all arrows are supported on $f^{-1}(\psi)$ components. Furthermore the other three arrows of equation (19), namely μQ, $\mu Q'$, and $[n]f^*(! \to *)1^!_\psi$, involve only $f^{-1}(\psi)$ components, and the other two objects of equation (19), $[n]f^*k_{S*}\mathbb{Q}$ and $[n]f^*k_{T*}\mathbb{Q}$ (we may assume $(! \to *)k_{Y!}\mathbb{Q} = k_{Y*}\mathbb{Q}$ for $Y \in \mathrm{Ob}\,(\mathcal{S})$), are supported on $\mathcal{X}'$. So it suffices to verify the commutativity of equation (19) when viewed on $\mathcal{X}'$, (obtained by base change of f via g, i.e., restricting to "ψ"), but by fiberwise n-dimensionality we have f' is strongly n-dimensional, and the commutativity is verified.

The case $Q = k_{S!}W$ and $Q' = k_{T!}W'$ follows similarly, as does the case for arbitrary ϕ, since any ϕ is a linear combination of $1^!_\psi$'s.

We claim the general commutativity in equation (19) now follows on all Q, Q' each of whose members is a direct sum of spaces $k_{S!}V$, since the morphisms in question operate componentwise and decompose according to direct summands; let us write this out in detail. A morphism $\phi\colon Q \to Q'$ is a collection of morphisms $\phi_i\colon Q_i \to Q'_i$ (such that $d'_i\phi_i = \phi_{i-1}d_i$ for all i, where d, d' are the differentials of Q, Q' respectively). By assumption,

$$Q_i = \bigoplus_{S\in \mathrm{Ob}(\mathcal{S})} k_{S!}W_{S,i}, \quad Q_i = \bigoplus_{S\in \mathrm{Ob}(\mathcal{S})} k_{S!}W'_{S,i}$$

for each i, and so each ϕ_i is the direct sum of

$$\phi_{i,S,S'} \in \mathrm{Hom}(k_{S!}W_{S,i}, k_{S'!}W'_{S',i}).$$

We wish to verify the commutativity of the diagram

$$\begin{array}{ccc} \bigoplus_{X,S=f(X)} k_{X*}(V_{X,j-n} \otimes W_{S,i}) & \xrightarrow{[-n](!\to*)f^*\phi} & \bigoplus_{X,S=f(X)} k_{X*}(V_{X,j-n} \otimes W'_{S,i}) \\ \mu Q \downarrow & & \downarrow \mu Q' \\ \bigoplus_S f^* k_S W_{S,i} & \xrightarrow{f^*(!\to*)\phi} & \bigoplus_S f^* k_S W'_{S,i} \end{array} \tag{22}$$

where the top row has double complexes viewed as complexes by, as usual, grouping along the diagonals. But

$$(\mu Q') \circ \big([-n](! \to *) f^* \phi\big)$$

decomposes into components

$$(\mu k_{S'} W'_{S',i}) \circ \big([-n](! \to *) f^* \phi_{i,S,S'}\big),$$

and similarly for

$$\big(f^*(! \to *)\phi\big) \circ (\mu Q)$$

into components

$$\big(f^*(! \to *)\phi_{i,S,S'}\big) \circ (\mu k_S W_{S,i}).$$

But we have seen that the components agree, hence the commutativity of equation (22).

Since every element of the derived category is isomorphic to a simple Q, we now appeal to general principles (see Paragraph 2.9.3).

□

10 The Base Change Morphism

In this section we study the "base change morphisms," natural maps $g^* f_* \to f'_* g'^*$ and $f'_! g'^* \to g^* f_!$ studied, in the context of the derived category, last section. We believe such a study may be useful in understanding various aspects of base change.

Consider a base change diagram (i.e., a Cartesian diagram):

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{f'} & \mathcal{S}' \\ g' \downarrow & & \downarrow g \\ \mathcal{X} & \xrightarrow{f} & \mathcal{S} \end{array}$$

There is a morphism of functors, $u\colon g^*f_* \to f'_*(g')^*$, known as the *base change morphism*, described in [SGA4.XII.4] (page 6); actually, two morphisms are described there, and later Deligne proves ([SGA4.XVII.2], with corrections to the proof given in [SGA4$\frac{1}{2}$], *Erratum pour SGA 4*) that they are the same morphism. It is built from $f_* \to f_*g'_*(g')^*$ (from the adjunctive morphism) which, since $f_*g'_* \simeq g_*f'_*$ (canonically), gives a map $f_* \to g_*f'_*(g')^*$ and therefore, by adjointness, $g^*f_* \to f'_*(g')^*$.

It is similarly possible to define a morphism $g^*\underline{\underline{R}}f_* \to (\underline{\underline{R}}f'_*)(g')^*$. We refer to this as the base change morphism for the derived category. If $(\underline{\underline{R}}f'_*)(g')^* = \underline{\underline{R}}(f'_*(g')^*)$, then this morphism results directly from the base change morphism[9].

We wish to know when the base change morphism is an isomorphim. It turns out that it is under a number of interesting conditions, including that f is strongly dimensional, but it is certainly not always true.

First we shall study base change for sheaves of the form $k_{P*}\mathbb{Q}$, where k_P is the one-point inclusion of $P \in \mathrm{Ob}\,(\mathcal{X})$.

First note that the map $\mathrm{Id} \to g'_*(g')^*$ of a sheaf F is given by the natural map

$$F(X) \to \varprojlim_{Z;g'(Z)\to X} F(g'(Z)).$$

Since for $F = k_{P*}\mathbb{Q}$ we have $F(Y) = \mathbb{Q}^{\mathrm{Hom}(P,Y)}$, in such a case the map $F \to g'_*(g')^*F$ is described as

$$\mathbb{Q}^{\mathrm{Hom}(P,Y)} \to \mathbb{Q}^M, \tag{23}$$

where

$$M = \varinjlim_{Z;g'(Z)\to X} \mathrm{Hom}(P, g'(Z)),$$

and where equation (23) arises out of a set theoretic map $L \to \mathrm{Hom}(P,Y)$. Next we apply f_*, obtaining $f_*F \to f_*g'_*(g')^*F \simeq g_*f'_*(g')^*F$, and finally obtaining $g^*f_*F \to f'_*(g')^*F$, each time only writing out the set theoretic maps that give rise to these morphisms. We easily see that the resulting set theoretic map for $(g^*f_*F)(Q) \to (f'_*(g')^*F)(Q)$ is

$$\varinjlim_{W;f'(W)\to Q} \mathrm{Hom}(P, g'(W)) \to \varinjlim_{Z;g(Z)\to g(W)} \varinjlim_{W;f'(W)\to Q} \mathrm{Hom}(P, g'(W))$$

[9]It is not hard to see that $\underline{\underline{R}}(v_*u^*)$ is not generally isomorphic to $(\underline{\underline{R}}v_*)u^*$; indeed take $v\colon L_1 \to \Delta_0$, and $u\colon L_1 \to \Delta_1$ (with u an isomorphism on objects). Could it be the case that we get an isomophism whenever $v = f'$ and $u = g'$ in a change of base diagram, always or under some reasonable condition?

$$\to \varinjlim_{W; gf'(W)\to Q} \mathrm{Hom}(P, g'(W)) \simeq \varinjlim_{W; fg'(W)\to Q} \mathrm{Hom}(P, g'(W))$$
$$\to \mathrm{Hom}\big(f(P), g(Q)\big)$$

So the surjectivity and injectivity (respectively) of the map $(g^* f_* F)(Q) \to (f'_*(g')^* F)(Q)$ is equivalent to the injectivity and surjectivity (respectively) of the natural map from

$$w\colon L = \varinjlim_{W; f'(W)\to Q} \mathrm{Hom}(P, g'(W)) \to \mathrm{Hom}\big(f(P), g(Q)\big). \tag{24}$$

This in turn amounts to factorizing morphisms $f(P) \to g(Q)$ by f of a morphism $P \to g'(W)$ followed by g of a morphism $f'(W) \to Q$. We introduce some terminology to make this precise.

Definition 10.1 *If $P \in \mathrm{Ob}\,(\mathcal{X})$ and $Q \in \mathrm{Ob}\,(\mathcal{S}')$ then a PQ-morphism is a morphism $\phi\colon f(P) \to g(Q)$; we say that a PQ-*factorization *of ϕ is a pair of morphisms $\nu \in \mathrm{Fl}\,(\mathcal{X})$ and $\mu \in \mathrm{Fl}\,(\mathcal{S}')$ such that ν has source P and μ has target Q and $\phi = (g\mu)(f\nu)$. We say that two PQ-factorizations of ϕ, (ν_1, μ_1) and (ν_2, μ_2) are primitively equivalent if there exist ν_{12}, μ_{12} such that $\nu_2 = \nu_{12}\nu_1$, $\mu_1 = \mu_2\mu_{12}$, and $f(\nu_{12}) = g(\mu_{12})$. We say that (ν_1, μ_1) and (ν_2, μ_2) are equivalent if they are equivalent under the transitive, reflexive closure of primitive equivalence.*

It is clear that in equation (24), a PQ-morphism, ϕ, is in the image of w iff it has a PQ-factorization. Furthermore, in this case the preimage of ϕ will be unique iff all PQ-factorizations are identified in the limit, L. But the inductive limit over (a diagram of) sets is simply the disjoint union of the sets modulo the equivalence relation that is the closure of identifying two elements of two sets if one is mapped to the other in the diagram. Thus the preimage of ϕ will be unique iff any two PQ-factorizations are equivalent. We conclude the following theorem.

Theorem 10.2 *Consider the base change morphism $u\colon g^* f_* \to f'_*(g')^*$ described above. Then u applied to the sheaf $k_{P*}\mathbb{Q}$ is injective (respectively, surjective) at the object $Q \in \mathcal{S}'$ iff each PQ-morphism has a PQ-factorization (respectively, any two PQ-morphisms are equivalent). Hence u is an isomorphism iff for each P, Q, every PQ-morphism has a PQ-factorization and any two PQ-factorizations of a morphism are equivalent.*

(The last sentence makes use of the fact that the triangular closure of the $k_{P*}\mathbb{Q}$ is all of $\mathbb{Q}(\mathcal{X})$.)

One can use this theorem to come up with conditions for which the base change morphism is injective and/or surjective. For example, if f is source liftable, it is easy to see that any PQ-morphism, ϕ, has a PQ-factorization (where the second morphism is id_Q), and any PQ-factorization is equivalent to one where the second morphism is id_Q. If f is precofibered (see [SGA1.VI.6.1]), then it is immediate that any two PQ-factorizations where the second morphism is id_Q are equivalent; however, we imagine the "unique up to equivalence" condition of the above theorem to be much weaker than that of being precofibered; precofibered implies that the PQ-factorizations with second morphism id_Q has a terminal element, whereas "unique up to equivalence" speaks of the connectedness of a (possibly larger) category.

To give an example of when the conditions of the theorem are not satisfied, take $\mathcal{S} = \Delta_1$ and f, g to be the inclusions of Δ_0 into 0 and 1 respectively (in this case $\mathcal{X}'$ is the empty category).

A Simple Remarks on Duality

In this section we try to generalize the setting of Ext duality and make some aspects of it more precise. The idea is that Hom of two objects of the category in question should carry some extra structure (such as that of a finite dimensional vector space) that has some notion of a dual.

In general there seem to be two approaches. First, one can speak about representability, adjoints, etc. with respect to the new structure of Hom, and hope this is compatible with the old (set theoretic) structure. This is what seems to be commonly done with Serre functors, and this is what we have done in the paper (and works fine). Another method is to lay down axioms regarding this new structure that ensure some level of compatibility; the more structure, the more automatic the compatibility becomes. We shall do this in two subsections: first we give what seems like minimal structure, but enough to talk about L $\to$ R (or Serre) functors; second we give enough structure that guarantees that representability means essentially the same thing.

A.1 Hom structures

Definition A.1 *Let $\mathcal{M}, \mathcal{V}$ be categories. To give $\mathcal{M}$ a $\mathcal{V}$-*Hom *structure is to give a functor*

$$\mathrm{Hom}^{\mathcal{V}}_{\mathcal{M}} \colon \mathcal{M}^{\mathrm{op}} \times \mathcal{M} \to \mathcal{V}$$

and a functor

$$\mathrm{forget} \colon \mathcal{V} \to (\mathrm{Ens})$$

(where (Ens) is the category of sets or elements of a universe) such that $\mathrm{forget} \circ \mathrm{Hom}^{\mathcal{V}}_{\mathcal{M}} \simeq \mathrm{Hom}_{\mathcal{M}}$.

In the above case one can assume that $\mathrm{forget} \circ \mathrm{Hom}^{\mathcal{V}}_{\mathcal{M}} = \mathrm{Hom}_{\mathcal{M}}$ by redefining $\mathrm{Hom}_{\mathcal{M}}$.

Definition A.2 *A* reversible category*, $\mathcal{V}$, is a category that is a $\mathcal{V}$-category with a contravariant functor, ι, to itself that is an essential involution, i.e.,* $\iota^2 \simeq \mathrm{Id}$.

Let $\mathcal{V}$ be a reversible category, and $\mathcal{M}$ a $\mathcal{V}$-category. For each $B \in \mathrm{Ob}\,(\mathcal{V})$ we define the *left-to-right* functor, $B^{\mathrm{L}\to\mathrm{R}}$ or $(\mathrm{L} \to \mathrm{R})B$ to be the functor

$$A \mapsto (\mathrm{forget})\iota\mathrm{Hom}^{\mathcal{V}}_{\mathcal{M}}(B, A);$$

as a function of B this functor $B^{\mathrm{L}\to\mathrm{R}}$ is clearly covariant functor. If $B^{\mathrm{L}\to\mathrm{R}}$ is representable for each B, then Yoneda's lemma shows that $(\mathrm{L} \to \mathrm{R})$ gives a (covariant) functor from $\mathcal{M}$ to itself; it is defined uniquely up to (unique) isomorphism and it is ambiguous up to isomophism. By abuse of notation we use $B^{\mathrm{L}\to\mathrm{R}}$ to denote the object representing $B^{\mathrm{L}\to\mathrm{R}}$. We say that the left-to-right functor is *$\mathcal{V}$-representable* if there is an isomophism

$$\mathrm{Hom}^{\mathcal{V}}_{\mathcal{M}}(A, B^{\mathrm{L}\to\mathrm{R}}) \simeq \iota\mathrm{Hom}^{\mathcal{V}}_{\mathcal{M}}(B, A)$$

that is natural in A.

The right-to-left functor is defined analogously, via the equation

$$\mathrm{Hom}^{\mathcal{V}}_{\mathcal{M}}(A^{\mathrm{R}\to\mathrm{L}}, B) \simeq \iota\mathrm{Hom}^{\mathcal{V}}_{\mathcal{M}}(B, A).$$

If both right-to-left and left-to-right functors are $\mathcal{V}$-representable, then $\mathrm{L} \to \mathrm{R}$ and $\mathrm{R} \to \mathrm{L}$ are quasi-inverses of each other, since

$$\mathrm{Hom}^{\mathcal{V}}_{\mathcal{M}}(A, B) \simeq \iota\mathrm{Hom}^{\mathcal{V}}_{\mathcal{M}}(B, A^{\mathrm{L}\to\mathrm{R}})$$

$$\simeq \iota^2 \mathrm{Hom}^{\mathcal{V}}_{\mathcal{M}}\big((A^{\mathrm{L}\to\mathrm{R}})^{\mathrm{R}\to\mathrm{L}}, B\big) \simeq \mathrm{Hom}^{\mathcal{V}}_{\mathcal{M}}\big((A^{\mathrm{L}\to\mathrm{R}})^{\mathrm{R}\to\mathrm{L}}, B\big),$$

and we may apply Yoneda's lemma and the functor (forget).

Let $\mathcal{M}_1, \mathcal{M}_2$ be categories with $\mathcal{V}$-Hom structure with $\mathcal{V}$ reversible. Consider a pair of functors, (F, G), with $F\colon \mathcal{M}_1 \to \mathcal{M}_2$ and $G\colon \mathcal{M}_2 \to \mathcal{M}_1$. We say that F is a left $\mathcal{V}$-adjoint to G (or G a right $\mathcal{V}$-adjoint to F) if there is an isomorphism of bifunctors

$$\mathrm{Hom}^{\mathcal{V}}_{\mathcal{M}_2}(FA, B) \simeq \mathrm{Hom}^{\mathcal{V}}_{\mathcal{M}_1}(A, GB)$$

in A and B. We easily see the following theorem (remarked in [BLL04], for vector spaces).

Theorem A.3 *Under the notation and assumptions of the previous paragraph, we have that G has a right adjoint*

$$(\mathrm{L} \to \mathrm{R})_{\mathcal{M}_2} F (\mathrm{R} \to \mathrm{L})_{\mathcal{M}_1},$$

provided the above left-to-right and right-to-left functors $\mathcal{V}$-exist.

Proof We have

$$\mathrm{Hom}_{\mathcal{M}_1}(GA, B) \simeq \mathrm{Hom}_{\mathcal{M}_1}\big((\mathrm{R} \to \mathrm{L})_{\mathcal{M}_1} B, GA\big) \simeq$$

$$\mathrm{Hom}_{\mathcal{M}_2}\big(F(\mathrm{R} \to \mathrm{L})_{\mathcal{M}_1} B, A\big) \simeq \mathrm{Hom}_{\mathcal{M}_2}\big(A, (\mathrm{L} \to \mathrm{R})_{\mathcal{M}_2} F (\mathrm{R} \to \mathrm{L})_{\mathcal{M}_1} B\big).$$

□

We conclude a similar theorem about the left adjoint of F. More generally assume the left-to-right and right-to-left functors are $\mathcal{V}$-representable in both $\mathcal{M}_1$ and $\mathcal{M}_2$. Then there is a sequence of left/right adjoints

$$\cdots, F_{-1}, G_{-1}, F = F_0, G_0, F_1, G_1, \cdots$$

where

$$F_i = (\mathrm{L} \to \mathrm{R})^i_{\mathcal{M}_2} F (\mathrm{R} \to \mathrm{L})^i_{\mathcal{M}_1}, \quad G_i = (\mathrm{R} \to \mathrm{L})^{-i}_{\mathcal{M}_1} G (\mathrm{L} \to \mathrm{R})^{-i}_{\mathcal{M}_2},$$

and where $(\mathrm{L} \to \mathrm{R}), (\mathrm{R} \to \mathrm{L})$ taken to a negative exponent means taking $(\mathrm{R} \to \mathrm{L}), (\mathrm{L} \to \mathrm{R})$ respectively to the corresponding positive exponent.

A.2 Representability

We finish this section by trying to make the above discussion a bit more satisfactory. Namely, in the above we spoke of $\mathcal{V}$-representability, whereas in practice it should follow "automatically" from representablility; similarly adjoints should always be $\mathcal{V}$-adjoints. This is clear in the case used in this paper, where $\mathcal{V}$ is the category of finite dimensional vector spaces and "forget" is the usual forgetful functor. Yet, we'd like to convince the reader that this automatic carrying over to the $\mathcal{V}$-structure can be done with some simple axioms that don't seem overly restrictive.

(Also note that $\mathcal{V}$-representability and $\mathcal{V}$-adjoints are all that are necessary to the previous subsection, to discuss $\mathrm{L} \to \mathrm{R}$ functors— representability, adjoints, and the functor "forget" are not necessary, but give the standard application.)

We fix notation as in the previous subsection, with $\mathcal{M}$ being a category with $\mathcal{V}$-Hom structure where $\mathcal{V}$ is reversible. We add the following axioms:

1. $\mathcal{V}$ has a unit, u, i.e., an object such that
$$\mathrm{Hom}^{\mathcal{V}}_{\mathcal{V}}(u, \,\cdot\,) \simeq \mathrm{Id}_{\mathcal{V}}$$
and
$$\mathrm{Hom}_{\mathcal{V}}(u, \,\cdot\,) \simeq \mathrm{forget};$$

2. the operation $A \Box B = \mathrm{Hom}^{\mathcal{V}}_{\mathcal{V}}(\iota A, B)$ has an associated natural isomorphism in A, B, C:
$$(A \Box B) \Box C \simeq A \Box (B \Box C)$$

3. both categories $\mathcal{M}, \mathcal{V}$ have "Hom compositions"
$$\mathrm{comp}^{\mathcal{V}}_{\mathcal{C}} \in \mathrm{Hom}^{\mathcal{V}}_{\mathcal{V}}\Big(\mathrm{Hom}^{\mathcal{V}}_{\mathcal{C}}(A, B), \mathrm{Hom}^{\mathcal{V}}_{\mathcal{V}}\big(\mathrm{Hom}^{\mathcal{V}}_{\mathcal{C}}(B, C), \mathrm{Hom}^{\mathcal{V}}_{\mathcal{C}}(A, C)\big)\Big)$$
for $\mathcal{C} = \mathcal{M}, \mathcal{V}$, natural in A, B, C, and "identities"
$$\mathrm{id}^{\mathcal{V}}_{A,\mathcal{C}} \in \mathrm{Hom}^{\mathcal{V}}_{\mathcal{V}}\big(u, \mathrm{Hom}^{\mathcal{V}}_{\mathcal{C}}(A, A)\big)$$
that satisfy the usual category identities and that restrict to their set-theoretic analogues upon applying the forgetful functor.

4. a $\mathcal{V}$-morphism is a monomorphism iff it is after applying "forget," and similarly for epimorphism and isomorphism.

A few remarks on these conditions. First, condition (2) is equivalent to

$$\mathrm{Hom}^{\mathcal{V}}_{\mathcal{V}}(A\Box B, C) \simeq \mathrm{Hom}^{\mathcal{V}}_{\mathcal{V}}\big(A, \mathrm{Hom}^{\mathcal{V}}_{\mathcal{V}}(B,C)\big),$$

the usual "exponential" condition. Note that some relations, such as $A\Box B \simeq B\Box A$ are automatic from the definition of ι. We don't know if condition (2) is absolutely necessary, but it certainly simplifies the discussion below; similarly for condition (4). We can speak of passing from a $\mathcal{V}$-morphism

$$\mathrm{Hom}^{\mathcal{V}}_{\mathcal{C}}(A,B)\Box\mathrm{Hom}^{\mathcal{V}}_{\mathcal{C}}(B,C) \to \mathrm{Hom}^{\mathcal{V}}_{\mathcal{C}}(A,C)$$

to one of sets

$$\mathrm{Hom}_{\mathcal{C}}(A,B) \times \mathrm{Hom}_{\mathcal{C}}(B,C) \to \mathrm{Hom}_{\mathcal{C}}(A,C)$$

by writing things as in condition (3) and applying the functor forget; similarly for any condition that can be written in terms of $\mathrm{Hom}^{\mathcal{V}}$'s and u's.

We finish by showing that representability in $\mathcal{V}$ or in (Ens) is the same. Clearly $\mathcal{V}$-representability implies (Ens)-representability.

Now let F be a contravariant $\mathcal{V}$-functor, meaning $F\colon \mathcal{M}^{\mathrm{op}} \to \mathcal{V}$ such that the functoriality is given by a map of vector spaces,

$$\mathrm{Hom}^{\mathcal{V}}_{\mathcal{M}}(A,B)\Box F(B) \to F(A),$$

for each A, B (natural in A, B). Assume that (forget)F is represented by M, i.e.,

$$(\mathrm{forget})F \simeq \mathrm{Hom}_{\mathcal{M}}(\,\cdot\,, M).$$

Then we claim that F is $\mathcal{V}$-representable by M. Indeed, we have Id_M corresponds to an element of (forget)$F(M)$ that corresponds to a $\mathcal{V}$-morphism $u \to F(M)$. We therefore get a map

$$\mathrm{Hom}^{\mathcal{V}}_{\mathcal{M}}(A,M) \simeq \mathrm{Hom}^{\mathcal{V}}_{\mathcal{M}}(A,M)\Box u \to \mathrm{Hom}^{\mathcal{V}}_{\mathcal{M}}(A,M)\Box F(M) \to F(A).$$

Upon applying "forget" we easily see that the above morphism $\phi_A\colon \mathrm{Hom}^{\mathcal{V}}_{\mathcal{M}}(A,M) \to F(A)$ maps to one direction of the isomophism between $\mathrm{Hom}_{\mathcal{M}}(A,M)$ and (forget)$F(A)$. It follows that ϕ_A is an isomorphism for each A, and therefore has a unique inverse, μ_A. This easily gives the desired $\mathcal{V}$-representability.

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