A Density Theorem for Purely Iterative Zero Finding Methods

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1 Introduction

The goal of this paper is to prove a theorem about the density of points for which a purely iterative root finding method converges to a root.

For $z \in \mathbf{C}$ and $f(z) = \sum_{i=0}^{d} a_i z^i$ consider a map

$$T_f(z) = \frac{P(z, f, f', \ldots, f^{(l)})}{Q(z, f, f', \ldots, f^{(l)})}$$

where P and Q are polynomials over $\mathbf{C}$. For each f, T_f is a map from $\mathbf{C} \cup \{\infty\}$ to itself which we think of as an iteration in a root finding method. We require that

1. $$T_f(z) = \frac{z^s P_0(f, zf', z^2 f'', \ldots)}{z^{s-1} Q_0(f, zf', z^2 f'', \ldots)} \tag{1.1}$$

   where P_0 and Q_0 are homogeneous polynomials of the same degree.

2. $T_f(z)$ depends only on z and the roots $r_1, \ldots, r_d$ of f, and

   $$A(T_f(z)) = T_{Af}(Az)$$

   for any linear map $A \colon z \mapsto az + b$, where

   $$Af(z) = a_d(z - Ar_1) \ldots (z - Ar_d)$$

for

$$f(z) = a_d(z-r_1)\dots(z-r_d).$$

3. $T_f(r) = r$, $|T_f'(r)| < 1$ for any root r of f.
4. $T_f(\infty) = \infty$, $|T_f'(\infty)| > 1$ for any f of degree > 1.

To measure the density of convergent points for T_f, let P_d denote the polynomials of degree d with roots in the unit ball. For a polynomial f, let

$$\Gamma_{T,f} = \{z : T_f^n(z) \to \text{ a root of } f \text{ as } n \to \infty\}$$

where T_f^n is the n-th iterate of T_f (i.e. $\Gamma_{T,f}$ is the set of points converging to a root of f under the iteration T_f). Let

$$A_{T,f} = |\Gamma_{T,f} \cap B_2(0)|\,.$$

Then $A_{T,f}/4\pi$ is the probability that a random point in $B_2(0)$ converges to a root.

Theorem 1.1 *Let T satisfy (1)-(4). Then for any d there is a $c > 0$ such that*

$$A_{T,f} > c \qquad \forall f \in P_d.$$

Furthermore, we have

$$A_{T,f,r} > c \qquad \forall f \in P_d, \qquad \forall r \text{ with } f(r) = 0,$$

where $A_{T,f,r}$ denotes the contribution to $A_{T,f}$ from the root r of f.

More precisely,

$$A_{T,f,r} = |\Gamma_{T,f,r} \cap B_2(0)|$$

where

$$\Gamma_{T,f,r} = \{z : T_f^n(z) \to r \text{ as } n \to \infty\}$$

The above density theorem was conjectured to hold for Newton's method by Smale in [Sma85]. This conjecture was proven in [Fri86]; the proof used some special properties of Newton's method and explicit bounds on the constants as a function of d were given. The above theorem applies to a much larger class of root finding methods, though no explicit bounds on c are given.

Examples of T satisfying (1)-(4) are

1. Newton's method, $T_f(z) = z - \frac{f}{f'}$.

2. Modified Newton's method, $T_f(z) = z - h\frac{f}{f'}$ with a constant h, $0 < h < 1$.

3. Taylor's method (see [Atk78])

$$T_f(z) = z + \sum_{i=1}^{k} \frac{d^i}{dt^i} \left(\frac{\phi_t(z)}{i!} \right) \Bigg|_{t=0} h^i$$

   where $\phi_t(z)$ solves

$$\frac{d\phi_t(z)}{dt} = -\frac{f(z)}{f'(z)}, \qquad \phi_0(z) = z$$

   with k a positive integer and h a positive number sufficiently small (depending on k).

4. Incremental Euler's method (see [Atk78])

$$T_f(z) = z + \sum_{i=1}^{k} \frac{(-hf(z))^k}{k!} g^{(k)}(f(z))$$

   with $g = f^{-1}$, k a positive integer, and h positive and sufficiently small.

5. Any iterate of a T satisfying (1)-(4). This shows that maps satisfying (1)-(4) may contain extraneous attractive fixed points. For example, Newton's method, even applied to polynomials of degree as low as three, can contain attractive periodic points of period two. Therefore the second iterate of Newton's method can have extraneous attractive fixed points.

To prove theorem 1.1, take any sequence $f_n \in P_d$; we will show that A_{T,f_n} can not approach 0 as n tends to infinity. By passing to a subsequence we can assume each of the f_n's coefficients, or equivalently each of f_n's roots, converge. If $A_{T,f}$ were continous in f at the limit of the f_n's then we would be done; the fact that $A_{T,f}$ can be discontious at f's having multiple roots makes the theorem more interesting. If the limit of f_n has at least one isolated root, one would get enough of a contribution to A_{T,f_n} from such an isolated root (for large n) to show that A_{T,f_n} is bounded away from zero. However, if all of f_n's roots tend to cluster into several groups as $n \to \infty$, we must look at the limiting geometry of each individual cluster to estimate A_{T,f_n}. So fix a cluster, and"blow-up" the picture of the roots at that cluster so that, while they remain in some bounded region, they separate into smaller subclusters. Again, if the blow-up of at least one cluster has as least one isolated root, we are done; the reason is that the isolated root's contribution to Γ_{T,f_n} contains a sequence of balls which, from the point of view of the subcluster, tend to ∞ and whose radii get larger. These balls, from the point of view of the original scale of the problem, look like a sequence of balls whose radii get smaller and smaller and whose center converges to the cluster's limit point. It is not hard to see that in the original scale of the problem, the largest ball is of appreciable size, thus bounding A_{T,f_n} from below.

If none of the clusters has an isolated root, we look at the geometry of each subcluster, blowing-up the picture at each subcluster. Since the blowing-up process separates a cluster of roots into at least two distinct subclusters, successive blowing-up eventually isolates the roots. One can then find balls in Γ_{T,f_n} for each root in the blow-up of the picture which isolates it, and back-up through the blow-ups until reaching the original scale of the problem, finding balls in each scale of blow-up which lie in Γ_{T,f_n}. The fact that this can be done for each root proves, in addition, the second part of the theorem.

The basic estimate for the existence of the aforementioned balls is lemma 2.2, proved in section 2. In section 3 we describe the blowing-up process more precisely, and show how lemma 2.2 can be applied backwards through the blowing-up process.

2 Some Preliminary Results

Let $g\colon N \to \mathbf{C} \cup \{\infty\}$ be a complex analytic map, for an open $N \subset \mathbf{C} \cup \{\infty\}$. Let $z \in N$ be a repelling fixed point, i.e. $g(z) = z$ and $|g'(z)| > 1$.

Lemma 2.1 *For any open $A \subset \mathbf{C}$ we have that for n sufficiently large,*

$$g^n \{B_\epsilon(z)\} \cap A \neq \emptyset.$$

Proof Apply Cauchy's formula for $(fg^n)'(z)$, where f is a Möbius function taking a point in A to ∞, and where the contour is a small circle around z.

For our maps T, we have that ∞ is a repelling fixed point so the lemma can be applied.

From condition (1)-(4) on T it is easy to see that

$$T_f'(\infty) = q(d) = \frac{Q_0(1, d, d(d-1), \ldots)}{P_0(1, d, d(d-1), \ldots)}$$

is a rational funciton of d independent of f, and that if r is a k-tuple root, then

$$T_f'(r) = \frac{P_0(1, k, k(k-1), \ldots)}{Q_0(1, k, k(k-1), \ldots)} = \frac{1}{q(k)}.$$

For any f we have that for z in a neighborhood of ∞,

$$T_f(z) = \frac{z}{q(d)} + O(1)$$

and

$$T_f'(z) = \frac{1}{q(d)} + O\left(\frac{1}{|z|}\right) \tag{2.1}$$

and T_f^{-1} is defined locally. We have

$$\frac{T_f(z)}{z} = \frac{1}{q(d)} + O\left(\frac{1}{|z|}\right)$$

and so for $|z|$ sufficiently large, we have $z_0 = z, z_{-1}, z_{-2}, \ldots$ given by $T_f(z_{-i}) = z_{-i+1}$ has $|z_{-n}|$ growing like $(q(d) - \epsilon)^n$ for any $\epsilon > 0$ depending on how large $|z|$ is, and thus

$$\begin{aligned} \frac{z_{-n}}{z} &= \prod_{i=0}^{n-1} \left(1 - O\left(\frac{1}{|z_{-i}|}\right)\right) q(d) \\ &= q^n(d)\left(1 - O\left(\frac{1}{|z|}\right)\right). \end{aligned} \tag{2.2}$$

The mean value theorem and equation 2.1 yield for, say, $r < |z|/2$,

$$T_f^n \left\{ B_{r'}(z_{-n}) \right\} \subset B_r(z) \tag{2.3}$$

with

$$\begin{aligned} r' &= r\, q^n(d) \left(1 - O\left(\frac{1}{|z_{-n+1}|} + \cdots + \frac{1}{|z|}\right)\right) \\ &= r\, q^n(d) \left(1 - O\left(\frac{1}{|z|}\right)\right). \end{aligned} \tag{2.4}$$

Let

$$\tilde{z} = \lim_{n\to\infty} \frac{z_{-n}}{q^n(d)},$$

the limit existing by virtue of equation 2.2. For any $r < |z|/2$, using equations 2.3 and 2.4, we have

$$T_f^n \left\{ B_{rq^n(d)/2} \left(\tilde{z} q^n(d)\right)\right\} \subset B_r(z) \tag{2.5}$$

for n sufficiently large (depending on r).

Next we would like to obtain a version of equation 2.5 for polynomials close to f in a certain sense. Fix d, D, and f, and consider the set $\mathcal{F}_{f,\delta,D}$ of polynomials

$$g(z) = (z - s_1) \dots (z - s_{d+D})$$

with $s_i \in B_\delta(r_i)$ for $1 \le i \le d$ and $|s_i| > 1/\delta$ for $i > d$.

Lemma 2.2 *For any sufficiently large z and $r < |z|/2$ there is a c, δ_0 and n_0 such that if $\delta < \delta_0$ and $n > n_0$ we have*

$$T_g^n \left\{ B_{rq^n(d)/2}(\tilde{z} q^n(d))\right\} \subset B_r(z)$$

if

$$|\tilde{z}| q^n(d) < \frac{c}{\delta}$$

for all $g \in \mathcal{F}_{f,\delta,D}$.

Proof Dividing both numerator and denominator by $z^{s-1}g^{\deg(P_0)}$ in condition (1) on T yields

$$T_g(z) = \frac{zP_0(1, z\frac{g'}{g}, z^2\frac{g''}{g}, \ldots)}{Q_0(1, z\frac{g'}{g}, z^2\frac{g''}{g}, \ldots)}.$$

For $|z|$ sufficiently large and, say, $\leq \frac{1}{2\delta}$ we have

$$\begin{aligned}\left|\frac{f'}{f} - \frac{g'}{g}\right| &\leq \sum_{i=1}^{d}\left|\frac{1}{z-r_i} - \frac{1}{z-s_i}\right| + \sum_{j=d+1}^{d+D}\left|\frac{1}{z-s_j}\right| \\ &= \sum\left|\frac{s_i - r_i}{(z-r_i)(z-s_i)}\right| + \sum\frac{1}{|z-s_i|} \\ &= O\left(\frac{\delta}{|z|^2} + \delta\right).\end{aligned}$$

Similarly we have

$$\begin{aligned}\left|\frac{f^{(k)}}{f} - \frac{g^{(k)}}{g}\right| &\leq \sum_{1\leq i_1<\ldots<i_k\leq d} k!\left|\frac{1}{(z-r_{i_1})\ldots(z-r_{i_k})} - \frac{1}{(z-s_{i_1})\ldots(z-s_{i_k})}\right| + \\ &\quad \sum_{1\leq i_1<\ldots<i_k\leq d+D,\ i_k>d} k!\left|\frac{1}{(z-s_{i_1})\ldots(z-s_{i_k})}\right| \\ &= O\left(\frac{\delta}{|z|^{k+1}} + \frac{\delta}{|z|^{k-1}} + \frac{\delta^2}{|z|^{k-2}} + \cdots + \delta^k\right) \\ &= O\left(\frac{\delta}{|z|^{k+1}} + \frac{\delta}{|z|^{k-1}}\right);\end{aligned}$$

in the last line we have used the fact that for sufficiently large z and n we have $|z|\delta < 1$ (which follows from the second equation in the statement of the lemma). Thus

$$\left|z^k\frac{f^{(k)}}{f} - z^k\frac{g^{(k)}}{g}\right| = O\left(\frac{\delta}{|z|} + \delta|z|\right)$$

and so

$$\begin{aligned}T_g(z) &= T_f(z)\left(1 + O\left(\frac{\delta}{|z|} + \delta|z|\right)\right), \qquad (2.6)\\ T_g'(z) &= T_f'(z)\left(1 + O\left(\frac{\delta}{|z|} + \delta|z|\right)\right).\end{aligned}$$

We caution the reader that the big-O notation above is as the quantity in parenthesis tends to zero, and that the constants in the big-O notation depend on d and D. Now fix a z sufficiently large and a small ϵ so that $z_0 = z, z_{-1}, z_{-2}, \ldots$ defined as before grow like a geometric series. Then, using equation 2.6, we see that for δ sufficiently small we have that $y_0 = z, y_{-1}, y_{-2}, \ldots, y_{-n}$ given by $T_g(y_{-i}) = y_{-i+1}$ grows like a geometric series, as long as $|y^{-n}| < c/\delta$ for c sufficiently small. Then we get

$$\begin{aligned} y_{-n} &= z_{-n} \prod_{i=0}^{n-1} \left(1 + O\left(\frac{\delta}{|y_{-i}|} + \delta |y_{-i}|\right)\right) \\ &= z_{-n} \left(1 + O\left(\frac{\delta}{|z|} + \delta |y_{-n}|\right)\right). \end{aligned}$$

Using the chain rule we have

$$\begin{aligned} (T_g^n)'(w) &= \prod_{i=0}^{n-1} T_g'(T_g^i(w)) \\ &= \left(\frac{1}{q(d)}\right)^n \left(1 + O\left(\frac{\delta}{|T_g^n(w)|} + \delta |w|\right)\right) \end{aligned}$$

assuming $|T_g^n(w)|$ is sufficiently large and $|w| \leq c/\delta$. The mean value theorem then implies

$$T_g^n \left\{B_{r'}(z_{-n})\right\} \subset B_r(z)$$

where

$$r' = r q^n(d) \left(1 + O\left(\frac{\delta}{|z|} + \delta |z_{-n}|\right)\right).$$

Hence, as before, we get that for sufficiently large n,

$$T_g^n \left\{B_{rq^n(d)/2}(\tilde{z} q^n(d))\right\} \subset B_r(z)$$

as long as $|\tilde{z}| q^n(d) < \frac{c}{\delta}$ for c sufficiently small.

3 Successive Normalizations

The difficulty in proving theorem 1.1 is that $A_{T,f}$ is not necessarily continuous when f has multiple roots. Let $f_1, f_2, \ldots$ be a sequence in P_d, and

$r_1^1, r_1^2, \ldots$ a sequence of respective roots for which

$$\lim_{n\to\infty} A_{T,f_n,r_1^n} = \inf_{f\in P_d,\, f(r)=0} A_{T,f,r}.$$

By passing to a subsequence we may assume that

$$f_n(z) = (z - r_1^n)^{e_1} \ldots (z - r_{k_0}^n)^{e_{k_0}}$$

with

$$e_1 + \cdots + e_{k_0} = d$$

and

$$r_i^n \neq r_j^n \qquad \forall n, \quad i < j \leq k_0.$$

By passing to a subsequence we can assume

$$r_i^n \to r_i \qquad \text{as } n \to \infty.$$

If r_1 is isolated, i.e. $e_1 = 1$, then it would be easy to show that for some $\delta > 0$ we have

$$B_\delta(r_i^n) \subset \Gamma_{T,f_n}$$

for all n sufficiently large, and thus

$$\inf_{f\in P_d,\, f(r)=0} A_{T,f,r} > 0$$

(the details of the argument appear as part of the proof later in this section). If not, we can assume

$$r_1 = r_2 = \ldots = r_{k_1}$$

and $r_j \neq r_1$ for $j > k_1$. We will now analyze more carefully the way in which $r_1^n, \ldots, r_{k_1}^n$ converge to r_1.

For $z_1, \ldots, z_m \in \mathbf{C}$ not all the same, we define the *normalization of* $z_1, \ldots, z_m$ *centered at* z_1 to be the unique linear map

$$g(z) = az + b, \qquad a \in \mathbf{R},\ a > 0,\ b \in \mathbf{C}$$

such that

$$\sum_{i<j} |g(z_i) - g(z_j)| = 1,$$

and $g(z_1) = 0$.

By passing to a subsequence we can assume that

1. the normalizations of $r_1^n, \ldots r_{k_0}^n$, $g_n(z) = a_n z + b_n$, centered at r_1^n have $g_n(r_i^n) \to s_i$ as $n \to \infty$, and

2. $$q_1^{\lfloor -\log_{q_1} a_n \rfloor} a_n \to a \tag{3.1}$$

   as $n \to \infty$ for some $a \in [1/q_1, 1]$ where

   $$q_1 = q\left(\sum_{i=1}^{k_1} e_i\right)$$

   and where $\lfloor \beta \rfloor$ denotes the largest integer $\leq \beta$.

Clearly

$$\sum_{i<j} |s_i - s_j| = 1,$$

and so we have

$$s_1 = \cdots = s_{k_2}$$

and $s_j \neq s_1$ for $j > k_2$ where $k_2 < k_1$. In other words, by normalizing we separate the first k_1 roots into smaller groups. By repeated normalization we will finally separate r_1^n from all other r_i^n's. Now we start with the deepest level of normalization and work up, proving a density lower bound for each level.

Let the deepest level be ℓ, and let

$$h_n(r_i^n) \to t_i \qquad \text{for } 1 \leq i \leq k_\ell$$

where h_n is the normalization of $r_1^n, \ldots, r_{k_\ell}^n$ centered at r_1^n. We have

$$\sum_{i<j} |t_i - t_j| = 1,$$

$t_1 = 0$, and $t_i \neq t_1$ if $i > 1$. Consider

$$\tilde{f}(z) = (z - t_1)^{e_1} \ldots (z - t_{k_\ell})^{e_{k_\ell}}.$$

Since $T_{\tilde{f}}(t_1) = t_1$, $|T'_{\tilde{f}}(t_1)| < 1$, and ∞ is a repelling fixed point for $T_{\tilde{f}}$ we have open sets E, arbitrarily near ∞, such that $T_{\tilde{f}}^n\{E\} \to t_1$ as $n \to \infty$. Take a point z large enough so that lemma 2.2 holds, with $B_\epsilon(z)$ converging to t_1 uniformly under $T_{\tilde{f}}$ for some $\epsilon > 0$ (we can assure uniform convergence by assumption (3) of section 1). We have

$$B_{\epsilon q_\ell^m/2}(\tilde{z} q_\ell^m) \subset \Gamma_{T,\tilde{f}}$$

for m sufficiently large where $\tilde{z}$ is as in lemma 2.2 and

$$q_\ell = q\left(\sum_{i=1}^{k_\ell} e_i\right).$$

Let h'_n be the normalization of the $\ell-1$-th level, i.e. of $r_1^n, \ldots, r_{k_{\ell-1}}^n$ centered at r_1^n,

$$h'_n(z) = a'_n z + b'_n$$

and let

$$h_n(z) = a_n z + b_n.$$

We have that

$$\frac{a_n}{a'_n} q_\ell^{\lfloor -\log_{q_\ell}(a_n/a'_n) \rfloor} \to a$$

as $n \to \infty$ for some $a \in [\frac{1}{q_\ell}, 1]$ (at each level we normalize and pass to a subsequence satisfying a condition analogous to that of equation 3.1 as well as the preceeding condition). We want to prove that

$$B_{\epsilon_0}(z_0) \subset \Gamma_{T,h'_n f_n} \tag{3.2}$$

for all sufficiently large n, where

$$\begin{aligned} z_0 &= \tilde{z} a q_\ell^{-M} \\ \epsilon_0 &= \epsilon a q_\ell^{-M}/4 \end{aligned}$$

for some positive integer M; this will complete the first stage moving backwards through the normalizations, each time finding a ball of fixed size with respect to the current normalization in Γ_{T,f_n} for sufficiently large n. To prove equation 3.2, consider first

$$h_n f_n(z) = (z - h_n(r_1^n))^{e_1} \ldots (z - h_n(r_{k_\ell}^n))^{e_{k_\ell}}.$$

We claim that for n sufficiently large we have

$$B_\epsilon(z) \subset \Gamma_{T,h_n f_n}.$$

To see this, we note that for some small $\eta > 0$ we have

$$|z - t_1| \leq \eta \Longrightarrow |T_{\tilde{f}}(z) - t_1| \leq (1-\mu)|z - t_1|,$$

by assumption (3) of section 1, for some $\mu > 0$, and that for some large N,

$$T_{\tilde{f}}^N \{B_\epsilon(z)\} \subset B_{\eta/2}(t_1)$$

by the uniform covnergence. Estimating as in lemma 2.2 (note that for any δ we have $h_n f_n \in \mathcal{F}_{\tilde{f},\delta,D}$ for n sufficiently large and $D = d - q_\ell$) we get that for n sufficiently large

$$\begin{aligned} |z - t_1| \leq \eta \quad &\Longrightarrow \quad |T_{h_n f_n}(z) - t_1| \leq (1 - \mu/2)|z - t_1| \\ &\Longrightarrow \quad z \in \Gamma_{T,h_n f_n} \end{aligned}$$

and that

$$T_{h_n f_n}^N \{B_\epsilon(z)\} \subset B_\eta(t_1) \subset \Gamma_{T,h_n f_n}$$

using $h_n(r_1^n) = t_1$ and that for any $y \in B_\epsilon(z)$ we have $y, T_{\tilde{f}}(y), T_{\tilde{f}}^2(y), \ldots$ stays away from the r_i^n's with $i > 1$. Now we apply lemma 2.2 to conclude that for m sufficiently large we have

$$T_{h_n f_n}^m \left\{B_{\epsilon q_\ell^m/2}(\tilde{z} q_\ell^m)\right\} \subset B_\epsilon(z) \subset \Gamma_{T,h_n f_n}$$

so that

$$B_{\epsilon q_\ell^m/2}(\tilde{z} q_\ell^m) \subset \Gamma_{T,h_n f_n}$$

as long as $|\tilde{z}| q_\ell^m < c/\delta$ for some c sufficiently small, where $1/\delta$ is a lower bound on $h_n(r_i^n)$ for $i > k_\ell$. Rescaling by a factor of a_n/a'_n and translating appropriately we get

$$B_{\epsilon q_\ell^m a_n/(2a'_n)}(\tilde{z} q_\ell^m a_n/a'_n) \subset \Gamma_{T,h'_n f_n}$$

if

$$|\tilde{z}| q_\ell^m a_n/a'_n < c \min_{i > k_\ell} h'_n(r_i^n) < c'. \tag{3.3}$$

Taking

$$m(n) = \lfloor \log_{q_\ell} \frac{a'_n}{a_n} \rfloor - M$$

where M is sufficiently large to ensure equation 3.3 holds, we get that for sufficiently large n,

$$B_{\epsilon a q_\ell^{-M}/4}(\tilde{z} a q_\ell^{-M}) \subset \Gamma_{T,h'_n f},$$

the 4 in $\epsilon a q_\ell^{-M}/4$ appearing to account for the fact that

$$\frac{a_n}{a'_n} q_\ell^{m(n)}$$

approaches, rather that equals, $a q_\ell^{-M}$ as $n \to \infty$. Thus equation 3.2 is established.

Now that we have a statement of the form

$$B_{\epsilon_0}(z_0) \subset \Gamma_{T,h'_n f_n},$$

we proceed to get a statement of the form

$$B_{\epsilon_1}(z_1) \subset \Gamma_{T,h''_n f_n},$$

where h''_n is the normalization at the $\ell - 2$-th level, i.e. the normalization of $r_1^n, \ldots, r_{k_{\ell-2}}^n$ centered at z_1^n. To do this we consider

$$\hat{f}(z) = (z - t_1)^{e_1} \ldots (z - t_{k_{\ell-1}})^{e_{k_{\ell-1}}}.$$

Using lemma 2.1 we can find an arbitrarily large z with an ϵ so that for some N

$$T_{\hat{f}}^N \left\{ B_{\epsilon/2}(z) \right\} \subset B_{\epsilon_0}(z_0).$$

Now we repeat the argument of before to conclude

$$T_{h'_n f_n}^N \left\{ B_\epsilon(z) \right\} \subset B_{\epsilon_0}(z_0)$$

i.e.

$$B_\epsilon(z) \subset \Gamma_{T,h'_n f_n}$$

(with uniform convergence) for n sufficiently large, and that

$$T_{h''_n f_n}^{m'(n)} \left\{ B_{\epsilon_1}(z_1) \right\} \subset \Gamma_{T,h''_n f_n}$$

(again with uniform convergence) for some $m'(n)$ and fixed ϵ_1, z_1.

Repeating the above argument $\ell - 2$ more times yields that for all n sufficiently large we have

$$B_\epsilon(z) \subset \Gamma_{T,f_n}$$

for some fixed ϵ and z with z very near r_1^n. Hence

$$\lim_{n\to\infty} A_{T,f_n,r_1^n} > \pi\epsilon^2 > 0$$

and theorem 1.1 is proven.

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