Math 441 - Class notes starting Oct. 10

- Good "Meaningless" Objective functions

Oct 10: Both progress reports & final project deadlines: one week more...

Some very useful objective functions do not have a "precise meaning".

Examples: - Scheduling group presentations via weighted matchings

- Markowitz model

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Say: 5 groups, 2 slots on MWF

Group 1

Group 2

⋮

Group 5

Day 1 ← M

Day 2 ← W

Day 3 ← F

We want "fair" scheme, e.g. (1) no two groups want to swap days, (2) if a group is assigned to some day, and there is an opening in a day they prefer, they get their perference, (3) etc.

Idea: Use weighted matching (!)

Group 1 : prefer : W, F, M

Weights 3 2 1

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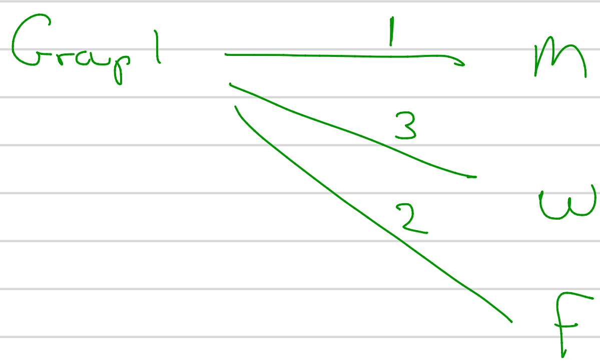
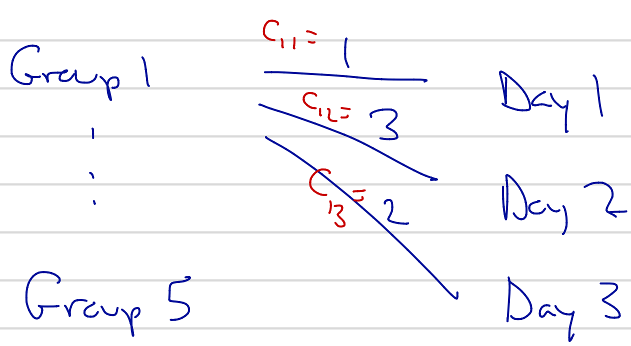


All groups give preferences M, W, F $\rightsquigarrow$ weights



$$X_{ij} \begin{cases} 1 & \text{if gp } i \to \text{day } j \\ 0 & \text{otherwise} \end{cases}$$

$i = 1, \ldots, 5$

$j = 1, \ldots, 3$

Then Utility : $\boxed{\max \sum c_{ij} x_{ij}}$ c_{ij} = Weights, expressing prefences

Group 1 assigned one Day: $X_{11} + X_{12} + X_{13} = 1$

⋮

Day 2 assigned ≤ 2 groups: $X_{12} + X_{22} + X_{32} + X_{42} + X_{52} \leq 2$ } sample

⋮

$X_{ij} \in \{0, 1\}$, or $X_{ij} \geq 0$, $X_{ij} \in \mathbb{Z}$

More precisely:

$$\text{group } i \quad \sum_{j=1}^{3} X_{ij} = 1, \qquad \text{day } j \quad \sum_{i=1}^{5} X_{ij} \le 2$$

C_{ij}: preferences

$$\max \quad \sum C_{ij} X_{ij}$$

← { ① no precise "thing" that it measures
② it produces a fair schedule ...

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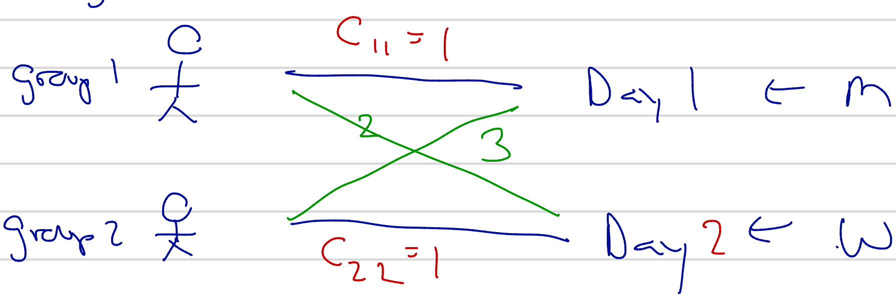
e.g.



group 1 F, W, M

group 2 M, F, W

Utility of $\begin{matrix} 1 \to 1 \\ 2 \to 2 \end{matrix}$ $= 1+1$, ← this can't be a utility maximizer

$1 \to 2,\ 2 \to 1$ $= 2+3 = 5$ larger

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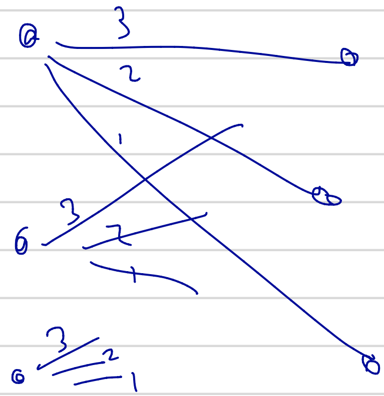
What if group 1 MWF preferences
group 2 MWF
⋮
group 5 MWF

If all preferences the same



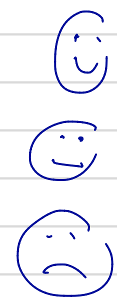
etc.

Gurobi will find some optimal solution:

2 groups M

2 groups W

1 group F



Fairer: Randomize the group names, then optimize $U = \sum c_{ij} x_{ij}$

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What if we want groups 1, 2, 3 get priority over groups 4, 5?

$$\text{Utility} \stackrel{?}{=} 1000\,\overbrace{(c_{11}x_{11}+c_{12}x_{12}+c_{13}x_{13}+\dots+c_{33}x_{33})}^{\sum_{i=1,2,3}\sum_j c_{ij}x_{ij}} \quad \text{grps } 1,2,3$$

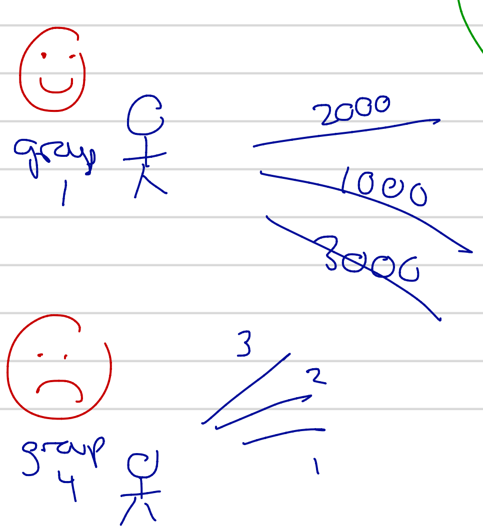
$$+\ \underbrace{(c_{41}x_{41}+c_{42}x_{42}+\dots+c_{53}x_{53})}_{} \quad \text{grps } 4,5$$

parameter

$1000\ U_1(x)$

$+$

$U_2(x)$



What if group 1 really doesn't like F,

" care between M, W ?

If M, W, F   W, M, F

weights 3 2 1   2 3 1

What if 3000 2000 1

could help...

OR

m	W	F
0	0	−1

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No difference

m	W	F
30	20	10

m	W	F
9	4	1

square of preferences ...?

Another scheme: m W F

anything that add up to 6 ? Want weights ≥ 0 ..

Group 1:

m	W	F
3	2	1

?

m	W	F
1003	1002	1001

Next time: Start Markowitz model

Oct 12: Proposal will be graded and returned via Canvas or email.

Most projects have identified an area.

Scores tend to be:

- An easy question
- Usually, have a parameter to play with

$U(\vec{x})$:

$U = U_1 + \alpha U_2$

parameter

→ A difficult

1. near 5/5 : You have identitified specific questions to work on.

Please book an appointment with me

near 2.5/5 : You have some ideas for specific questions.

near 0/5 : You have an area but no specific questions.

You can resubmit the proposal for [more points]

a reasonable direction

Oct 10: Markowitz Model: (Quadratic Program)

Bottom Line: You can hold $w_1, \dots, w_n$ (each real, ≥ 0) shares of financial instruments $R_1, \dots, R_n$.

Utility$(w_1, \dots, w_n ; \mu)$

positive parameter
risk tolerance/aversion

$$= \underbrace{\text{Expected Return}}_{\text{linear function of } w_1,\dots,w_n} - \mu \underbrace{\underbrace{\text{Variance}}_{\substack{\text{Quadratic function} \\ \text{of } w_1,\dots,w_n \\ (\text{ALWAYS } \geq 0 \text{ !!})}}}_{\text{not the best measure of risk}}$$

(1) We will review expected value & variance.

(2) I will convince you that $U_{\text{Markowitz}}$ is not always realistic

(3) " " " " " " is useful in practice.

Eventually will study §5, page 4:

	Event 1	Ev 2	Ev 3	Ev 4	avg	var
Prob, p	1/4	1/4	1/4	1/4		
C (Coke)	8	8	12	12	10	4
P (Pepsi)	8	8	12	12	10	4
A (Amazon)	8	12	8	12	10	4
N (Nozama)	12	8	12	8	10	4
B (Bond)	9	9	9	9	9	0

(from my article). Start with

$$L = \text{lottery ticket} \begin{cases} \text{pays } \$10 & \text{with prob } \frac{1}{10} \\ \text{pays } 0 & \text{" " } \frac{9}{10} \end{cases}$$

$$M = \text{a "riskier" lottery ticket} \begin{cases} \text{pays } \$1000 & \text{" " } \frac{1}{1000} \\ \text{pays } 0 & \text{" " } \frac{999}{1000} \end{cases}$$

What is expected value & Variance of

L, $2L$, $L + L'$

M, etc.

"Review"

$$L_1 = 10$$

$$L = \begin{cases} \text{pays } \$10 & \text{prob } p_1 = \frac{1}{10} \\ \text{pays } \$0 & \text{" } p_2 = \frac{9}{10} \end{cases}$$

$$L_2 = 0$$

In general: $p_1, \ldots, p_n \geq 0$, $p_1 + \ldots + p_n = 1$

If event i, prob p_i has value L_i,

(expected value)(L) = avg = $p_1 L_1 + \ldots + p_n L_n$

$= \bar{L} = \mathbb{E}(L)$ here $p_1 L_1 + p_2 L_2 = \frac{1}{10} \cdot 10 + \frac{9}{10} \cdot 0 = 1$

Variance (L) = Expected Value $(L - \bar{L})^2 = \mathbb{E}(L - \bar{L})^2$

$= \overline{(L - \bar{L})^2}$

$= p_1 (L_1 - \bar{L})^2 + \ldots + p_n (L_n - \bar{L})^2 =$

here $\bar{L} = 1 = \mathbb{E}(L)$, $\mathrm{Var}(L) = \frac{1}{10}(10-1)^2 + \frac{9}{10}(0-1)^2$

$= \frac{1}{10} \cdot 81 + \frac{9}{10} \cdot 1 = \frac{81+9}{10} = \frac{90}{10} = 9$

What does $\mathrm{Var}(L)$ mean?

① $\mathrm{Var}(L) \geq 0$

② $\mathrm{Var}(L) = 0 \implies L$ has a constant value

③ $\mathrm{Var}(2L) = 4\,\mathrm{Var}(L)$

$$L = \begin{cases} 10 & \text{prob } p_1 = \frac{1}{10} \\ 0 & \text{prob } p_2 = \frac{9}{10} \end{cases}, \qquad 2L = \begin{cases} 20 & \text{---} \\ 0 & \text{---} \end{cases}$$

$$\mathbb{E}[2L] = 2\,\mathbb{E}[L]$$

$$\mathrm{Var}(2L) = \frac{1}{10}(20-2)^2 + \frac{9}{10}(0-2)^2$$

$$4\left[\frac{1}{10}(10-1)^2 + \frac{9}{10}(0-1)^2\right] = 4\,\mathrm{Var}(L)$$

$$\mathrm{Var}(2L) = \mathbb{E}\left(2L - \overline{2L}\right)^2 = \mathbb{E}\left[2^2(L-\bar{L})^2\right] = 4\,\mathbb{E}(L-\bar{L})^2$$

Take two lottery tickets, L, L'

$$\mathbb{E}[L+L'] = \mathbb{E}[L] + \mathbb{E}[L']$$

(formally: $\quad p_1 \begin{Bmatrix} L_1 \\ L_1' \end{Bmatrix}, \ldots, p_n \begin{Bmatrix} L_n \\ L_n' \end{Bmatrix}$

$$\mathbb{E}[L+L'] = p_1(L_1+L_1') + \ldots + p_n(L_n+L_n')$$

$$= \underbrace{(p_1L_1 + p_2L_2 + \ldots + p_nL_n)}_{\mathbb{E}[L]} + \underbrace{(p_1L_1' + p_2L_2' + \ldots + p_nL_n')}_{\mathbb{E}[L']}$$

$$\mathbb{E}[aL] = a\,\mathbb{E}[L]$$

$a \in \mathbb{R}$

$$\mathrm{Var}(L+L') = \mathbb{E}\left[L+L' - \overline{L+L'}\right]^2$$

$$= \mathbb{E}\left((L-\bar{L})+(L'-\bar{L'})\right)^2$$

$$= \mathbb{E}\left((L-\bar{L})^2+(L'-\bar{L'})^2+2(L-\bar{L})(L'-\bar{L'})\right)$$

$$= \mathrm{Var}[L] + \mathrm{Var}(L') + 2\,\mathrm{Cov}(L,L')$$

$\mathrm{Cov}(L,L')$ = Means what??

definition $\mathbb{E}\left((L-\bar{L})(L'-\bar{L'})\right)$

=

$\mathrm{Cov}(L,L) = \mathrm{Var}(L)$

$\mathrm{Cov}(L,-L) = -\mathrm{Var}(L)$

$\mathrm{Cov}(L,L') = 0$ $\begin{cases} \text{happens if } L, L' \text{ "independent"} \\ \text{more generally "uncorrelated"} \end{cases}$

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$\mathrm{Var}(L+L')$, if $L'=L$, $\mathrm{Var}(2L) = 4\,\mathrm{Var}(L)$

$\mathrm{Var}(L+L')$, if $\mathrm{Cov}(L,L')=0$, $\mathrm{Var}(L+L') = \mathrm{Var}(L)+\mathrm{Var}(L')$

Oct 15: Markowitz model:

parameter positive

$$\text{Utility}(R) = \overline{R} - \mu \underbrace{\text{Var}(R)}_{\text{Var, CoVar}}$$

"Meaningless" but useful

Goal:

From article online

	Event 1	Ev 2	Ev 3	Ev 4	avg	var
Prob, p	1/4	1/4	1/4	1/4		
C (Coke)	8	8	12	12	10	4
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N (Nozama)	12	8	12	8	10	4
B (Bond)	9	9	9	9	9	0

Warmup:

$$L = \begin{cases} \$10 & \text{prob } p_1 = \frac{1}{10} \\ \$0 & \text{prob } p = 9/10 \end{cases}, \quad M = \begin{cases} \$1000 & \text{prob } \frac{1}{1000} \\ 0 & \text{prob } \frac{999}{1000} \end{cases}$$

Expect: \$1 for L, M: Expected \$1

Want to understand:

① $\overline{R} = E(R) =$ expected value (R),

② $\text{Var}(R) = \text{Variance}(R)$

③ $\text{Cov}(R, S) = E\left((R-\overline{R})(S-\overline{S})\right)$

Really a cosine

④ $\text{Corr}(R,S) = \dfrac{\text{Cov}(R,S)}{\sqrt{\text{Var}(R)\,\text{Var}(S)}}$ between -1 and 1

Lottery tickets:

$$L = \begin{cases} 10 & \text{prob } p_1 = \frac{1}{10} \\ 0 & \text{prob } p_2 = \frac{9}{10} \end{cases}$$

$$E[L] = \overline{L} = 10\,p_1 + 0\,p_2$$
$$= 10 \cdot \frac{1}{10} + 0 \cdot \frac{9}{10} = 1$$

$$\text{Var}(L) = \underline{E\,(L - \overline{L})^2}$$
$$= E\,(L - 1)^2 \qquad \text{(always non-negative)}$$
$$= (10-1)^2 \cdot \frac{1}{10} + (0-1)^2 \cdot \frac{9}{10}$$
$$= 81 \cdot \frac{1}{10} + 1 \cdot \frac{9}{10} = \frac{90}{10} = 9$$

$$M = \begin{cases} 1000 & \text{prob } \frac{1}{1000} \\ 0 & \text{prob } \frac{999}{1000} \end{cases} \qquad E(M) = 1$$

$$\text{Var}(M) = E\,(M-1)^2 = (1000-1)^2 \frac{1}{1000} + (0-1)^2 \cdot \frac{999}{1000}$$
$$= 999^2 \cdot \frac{1}{1000} + 1 \cdot \frac{999}{1000}$$
$$= 999\,(999+1)\,\frac{1}{1000} = 999$$

$$Q = \begin{cases} B & \text{prob } \frac{1}{B} \\ 0 & \text{otherwise} \end{cases}, \quad \overline{Q} = 1, \quad \text{Var} = B - 1$$

Extreme $B = 1$ $\quad \text{Ticket} = \begin{cases} 1 & \text{prob } \frac{1}{1} \\ 0 & \text{prob } 1 - \frac{1}{1} = 0 \end{cases}, \quad \text{Var} = 1 - 1 = 0$

units $(P-\bar{P})^2$

$\$$ CAD/year $\qquad$ $(\$\text{CAD})^2/\text{year}$

Markowitz Utility:

$$\text{Utility}_{\mu}(\text{Portfolio}) = \mathbb{E}[P] - \mu \,\text{Var}(P)$$

$\frac{1}{\$\text{CAD}}$

Claim: For any $\mu > 0$, there's a lottery ticket with negative utility...

E.g. $\mu = 0.01 = \frac{1}{100}$

Lottery ticket M $\begin{cases} 1000 & \text{prob } \frac{1}{1000} \\ 0 & \text{prob } \frac{999}{1000} \end{cases}$: $\bar{M} = 1$, $\text{Var}(M) = 999$

$$\text{Util}_{\text{Mark}, \mu=\frac{1}{100}} = 1 - \frac{1}{100}\, 999$$

$= \text{negative}$

So Markowitz Utility $\Rightarrow$ throw away the ticket

And yet, Markowitz is useful in practice...

Units of μ in Mark. obj. a bit strange...

$$\text{Util}_{\text{Mark}, \mu}[R] = \mathbb{E}(R) - \mu \,\text{Var}(R) :$$

claim: maximizing it produces reasonable optimal solutions to study...

Examples:

Four possible futures in the world

From article online

	Event 1	Ev 2	Ev 3	Ev 4	avg	var
Prob, p	1/4	1/4	1/4	1/4		
C (Coke)	8	8	12	12	10	4
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B (Bond)	9	9	9	9	9	0

? ??

Bond B: $\bar{B} = 9$, $Var(B) = E(B-\bar{B})^2 = 0$

Coke C: $\bar{C} = 10$ $Var(C) = E(C-10)^2$

$= E(4) = 4$

$\bar{P}, \bar{A}, \bar{N} = 10$.

$A+N = 20$ in all events. $\overline{A+N} = 20$

$Var(A+N) = 0$ Unrealistic

$\underbrace{Corr(A,N) = -1}$ (perfect hedge)