MATH 441

Notes Starting Sept 12

Sept 12: • Most material next 2 weeks: article applications.pdf on class website

Plan: • Give some standard LP and IP (integer programming) applications.

• Later we'll divide the room by LP, IP, etc. applications to help people find groups.

• No class on Wednesday, Sept. 19.

LP apps: Resource Allocation
- Scarce resources
- Expensive resources

Tasks with Wait Times

Mention others: Matrix games

Surprise LP appt:

- Some piecewise-linear programs ← (mathematical program)
- Weighted bipartite matching ← (IP)

IP apps:
- Bin packing

Bounded resources

Scarce resource example:

Resources
- Wood
- Labour

Products
- Tables
- Chairs

Objective

Maximize $5(\#\text{tables}) + 3(\#\text{chairs})$

Table:	3 units of wood,	1 hours of labour
Chair:	1 unit of wood,	2 hours of labour
You have	100 units wood,	120 hours of labour

Quest: Write LP; is it feasible and bounded; what else can we say?

x_T, x_C = # of tables, chairs

$$\max z = 5x_T + 3x_C$$

$$\text{s.t. } 3x_T + x_C \leq 100$$

$$x_T + 2x_C \leq 120$$

$$x_T, x_C \geq 0$$

5 means $ per table

3 means unit of wood per table

[for now $x_T, x_C \in \mathbb{R}$ (reals) but maybe $x_T, x_C \in \mathbb{Z}$ integers]

Here $\vec{x} = \begin{bmatrix} x_T \\ x_C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ feasible.

In these problems: $\max \vec{c} \cdot \vec{x}$

$$\text{s.t. } A\vec{x} \leq \vec{b}$$

$$\vec{x} \geq \vec{0}$$

where entries of A and components of $\vec{c}, \vec{b}$ are non-negative.

Typically feasible and bounded for simple reasons...

- Dual problem & original LP have "economic interpretation"

Expensive resources:

Apples: \$0.2 each, Milk: \$0.70 each litre.

x_1 x_2

Require: $x_1 + x_2 \geq 10$ (carb)

$x_1 + 8x_2 \geq 30$ (protein)

Quest: Write LP; is it feasible and bounded; what else can we say?

$$\begin{cases} \text{minimize} & (.2)x_1 + (.7)x_2 \\ \text{maximize} & -(.2)x_1 - (.7)x_2 \end{cases}$$

$10 \leq x_1 + x_2$ (standard form) $-x_1 - x_2 \leq -10$

$30 \leq x_1 + 8x_2$ " " $-x_1 - 8x_2 \leq -30$

$x_1, x_2 \geq 0$ (maybe $x_1, x_2 \in \mathbb{Z}$, maybe not, maybe $\mathbb{R}$)

This LP is feasible: take x_1, x_2 very large
LP is bounded: min has to be ≥ 0...

But $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ not feasible...

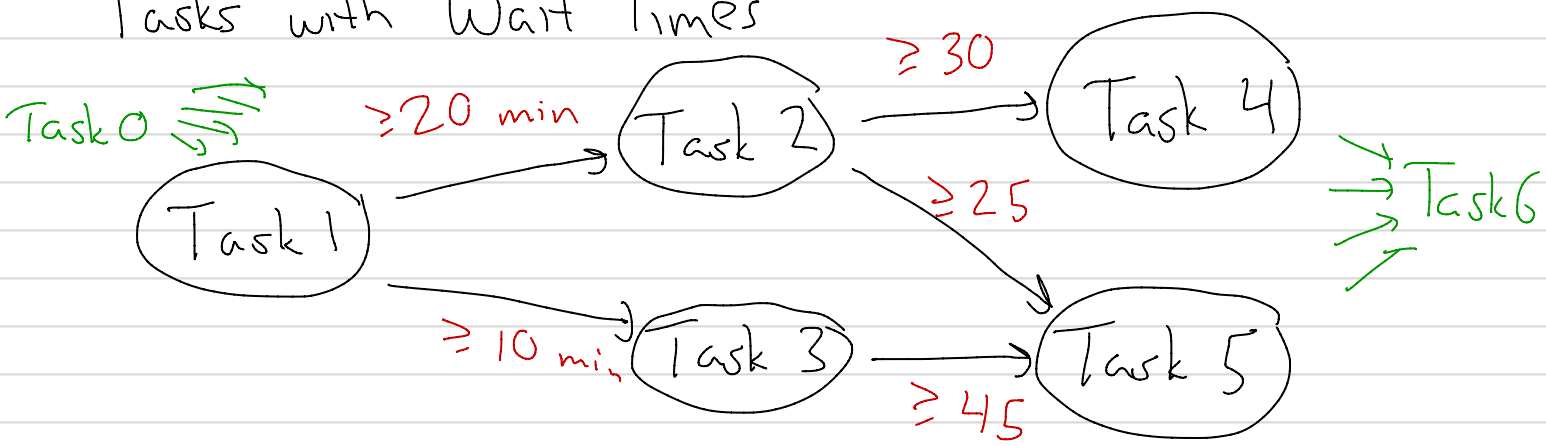
$\max \vec{c} \cdot \vec{x}$
s.t. $A\vec{x} \leq \vec{b}$
$\vec{x} \geq 0$

here A's entries and all of $\vec{b}, \vec{c}$ components are negative...

Dual LP:
$\vec{c} \to -\vec{b}$
$\vec{b} \to -\vec{c}$
$A \to -A^T$

These diagrams shouldn't have cycles

Tasks with Wait Times



✱ Tasks take little time to perform, but

- Task 2 must be done at least 20 min after Task 1
- Task 3 at least 10 min after Task 1
- etc.

$=$

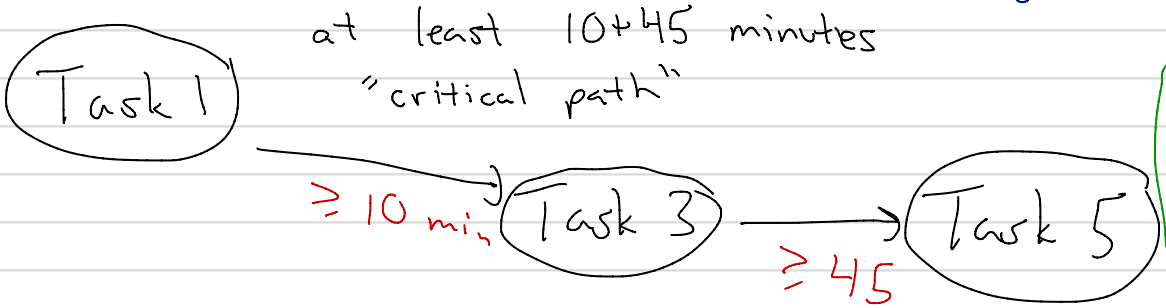
x_i time we do task i:

$\min \; x_6 - x_0$ s.t.

$x_0 \le x_1, \; x_0 \le x_2, \ldots,$
$x_6 \ge x_5, x_4, x_3, \ldots$

$x_1 + 20 \le x_2$
$x_1 + 10 \le x_3$
$\vdots$
$x_2 + 25 \le x_5$
$x_3 + 45 \le x_5$

Rem: "Path" implies need at least 10+45 minutes "critical path"



"Critical path" picked out by dual LP

Also $x_6 - x_0 = (x_6 - x_5) + (x_5 - x_3) + (x_3 - x_1) + (x_1 - x_0) \ge 0 + 45 + 10 + 0 \ge 55$
$\ge 0 \quad \ge 45 \quad \ge 10 \quad \ge 0$

Sept 14: - Homework due "Sept 21" really Sept 23, 11:59pm
- New webpage on Fundamentals Unix & Gurobi

Parametric LP

$$\max \; z = 10x_1 + 9x_2 + Ax_3 \quad \text{s.t.} \quad (A \in \mathbb{R} \text{ is a parameter})$$

$$x_1 + x_2 + x_3 \leq 12. \quad (\text{all } x_i\text{'s interchangeable})$$

$$x_1, x_2, x_3 \geq 0$$

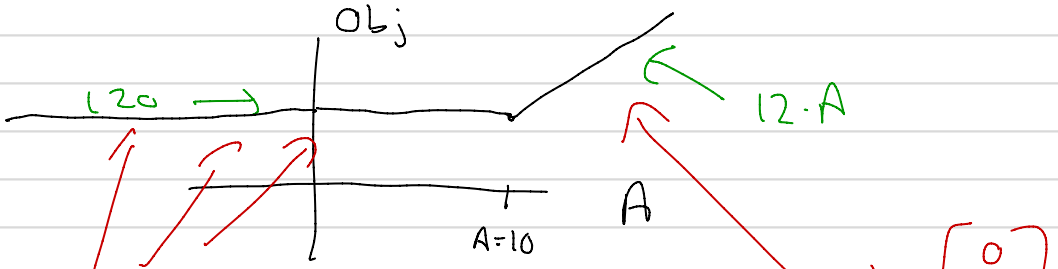
Two cases:

$A > 10$: Opt Sol $x_3^* = 12$, $x_2^* = x_1^* = 0$ unique

(Optimum) Objective $10 \cdot 0 + 9 \cdot 0 + A \cdot 12 = 12A$

$A < 10$: Opt Sol $x_1^* = 12$, $x_2^* = x_3^* = 0$

(Optimum) Objective $= 10 \cdot 12 + 9 \cdot 0 + A \cdot 0 = 120$



$x^* = \begin{bmatrix} 12 \\ 0 \\ 0 \end{bmatrix}$ $x^* = \begin{bmatrix} 0 \\ 0 \\ 12 \end{bmatrix}$

$A = 10$, $x^* \overset{??}{=} \begin{bmatrix} 12 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 12 \end{bmatrix}, \begin{bmatrix} 9 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} t \\ 0 \\ 12-t \end{bmatrix}$ $0 \leq t \leq 12$

SIMPLEX METHOD RETURNS

Sept 17:

- No class on Wednesday (Sept 19)
- Unix, DOS (Windows), Gurobi basics: new webpage
- Math 441 published as of today on canvas.ubc.ca
- Homework #1 due on Sunday, 11:59 pm
    - Try submitting something to test the system before Sunday
    - Submit one PDF file for the entire homework

Seems Difficult since 1970's

Bin Packing (IP) Integer Programming

A project has 5 tasks taking 20, 13, 12, 7, and 3 hours. Each task has to be done by one person. What is the longest amount of time that a person must work if you project has 3 people?

$x_i = 1$ if person 1 does task i.
0 otherwise

X $x_1, \dots x_5$

Y $y_1, \dots, y_5$ $y_i = 1$ " " 2 " " i.

Z $z_1, \dots, z_5$ $z_i = 1$ " " 3 " " i.

min w (max $-w$) subject to

$20x_1 + 13x_2 + 12x_3 + 7x_4 + 3x_5 \le w$

$20y_1 + 13y_2 + 12y_3 + 7y_4 + 3y_5 \le w$

$20z_1 + 13z_2 + \dots + 3z_4 \le w$

$x_1 + y_1 + z_1 = 1$ (Task 1)

$x_2 + y_2 + z_2 = 1$ (Task 2)

$\vdots$

And

$0 \le x_1, \dots x_5, y_1, \dots, y_5, z_1, \dots z_5 \le 1$

and are in $\mathbb{Z}$

$(x_1, \dots, z_5 \in \{0, 1\}$ Boolean$)$

Integer Program

Bin packing:

Tasks length: 200, 19, 17, 26, 53, 2, 7, 98

How to divide among 3 people s.t. max work that any one person does is minimized?

In practice: sort: 200, 98, 53, 26, 19, 17, 7, 2

Step 1: 200 → X

Step 2: next largests → Y until go over 200

etc:

Say: 200, 199, 198, etc...

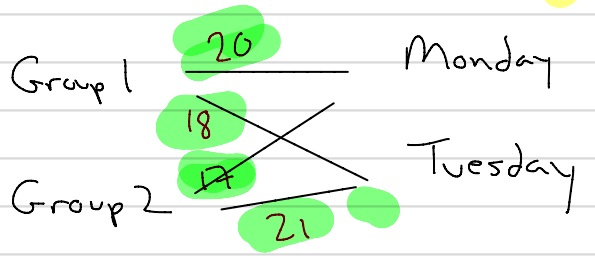
X → 200,

Y → 199, 198,

Often there are practical algorithms that work well to within some reasonable % of error...

Weighted Bipartite Matching (an IP that can be solved with LP)

(if you use simplex method)
(or if opt. sol. is unique)



$$\text{Utility} = 20X_{1M} + 18X_{1T} + 17X_{2M} + 21X_{2T}$$

$$X_{1M} = \begin{cases} 1 & \text{if Group 1 presents on Monday} \\ 0 & \text{otherwise} \end{cases}$$

Constraints:

Group 1: $X_{1M} + X_{1T} = 1$

Group 2: $X_{2M} + X_{2T} = 1$

Monday: $X_{1M} + X_{2M} = 1$

Tuesday: $X_{1T} + X_{2T} = 1$

X_{1M}, X_{1T}
X_{2M}, X_{2T}
0 or 1 ---

$0 \le$ all vars ≤ 1

all vars are integers

Maximize Utility s.t.

solves the matching problem

Sept 21, 2018

Deadline for proposals ⇝ Friday, October 5

Applications: - Graph Colouring
- "Baby" Sudoku (4×4 Sudoku)
- Markowitz Model, Traveling Salesman Problem (TSP),
- Progressive Taxation

Convex programming

Homework #1 is "out of 20 points" 20 might change

Problems with software:

- Gurobi on Mac you might be missing Xcode Command Line tools
- Lindo (may have used in Math 340)

  INTEGERS in LINDO means variable $\{0,1\}$

  GIN ("generalized integer") means integer ($\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$)

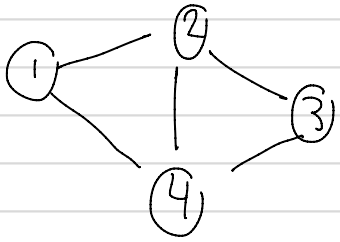
Graph Colouring:

Graph :

Vertex Set

Edge Set = set of "pairs" of vertices

example

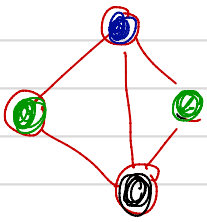


$V = \{1, 2, 3, 4\}$

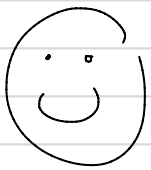
$E = \{\{1,2\}, \{2,3\}, \{3,4\}, \{4,1\}, \{2,4\}\}$

Problem: Given a graph $V = \{1, \ldots, n\}$ (given n) given E, given a number of colours. Say 3 colours. Question: can you colour the vertices so that no two vertices of the same colour are joined by an edge.

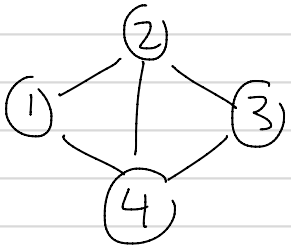
Example



this graph, 3 colours:



yes



3 colours. $= \{1,2,3\}$

Our intuition

Let $x_{ij} = \begin{cases} 1 & \text{if vertex } i \text{ is coloured } j \\ 0 & \text{otherwise} \end{cases}$

Is it feasible that:

x_{ij} are $\{0,1\}$, $i = 1,\dots,4$, $j = 1,2,3$

$0 \le x_{ij} \le 1$, $x_{ij} \in \mathbb{Z}$ = integers :

Vertex 1 should have one colour: $x_{11} + x_{12} + x_{13} = 1$

2

3 similarly

4

Vertex 1 & 2 should be different colours: $x_{11} + x_{21} \le 1$

$x_{12} + x_{22} \le 1$

$x_{13} + x_{23} \le 1$

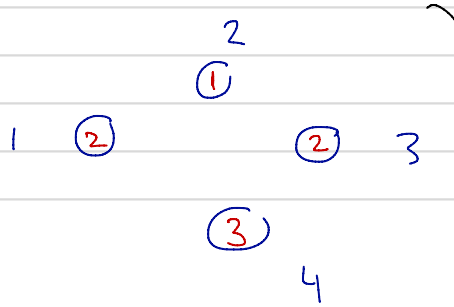
2 & 3

2 & 4

1 & 4 similarly

3 & 4

There exists a 3 colouring iff this IP is feasible



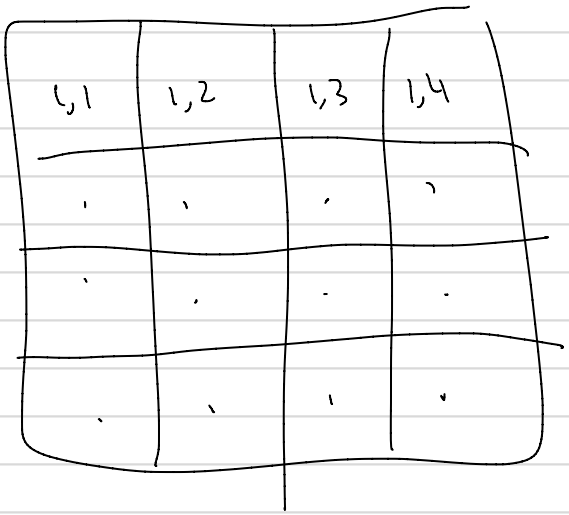
$x_{12} = 1$, $x_{11} = 0$, $x_{13} = 0$

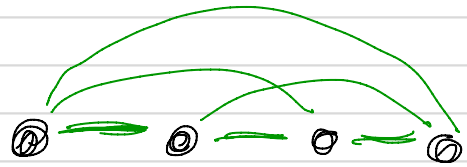
$x_{21} = 1$, $x_{22} = 0$, $x_{23} = 0$

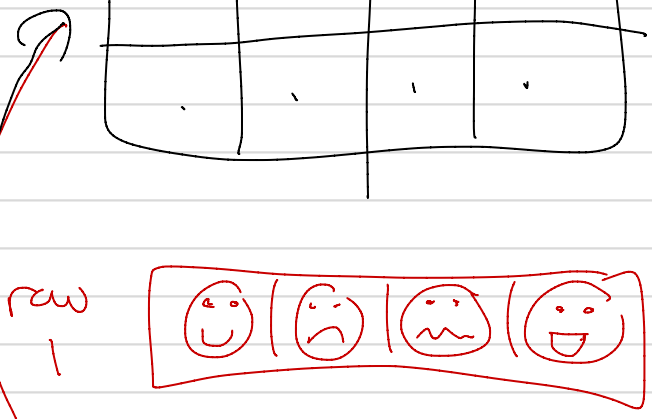
etc.

Sudoku 4×4 (easy)

4 colours {1,2,3,4}







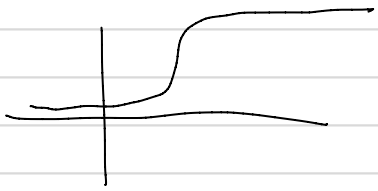
row 1

OR

14	2	3	4
5	6	7	8
9	10	11	12
13	16	15	1

Sept 24:

- Wednesday: latter part of class devoted to finding project groups
- Next Homework: Magic square type question
  (Rem: Gurobi has a Sudoku lp file / model example)
- Threshold phenomena are always a good (reliable) research topic:



rather than

- Sample proposal will be on website

- A few other IP (integer programs)
- Quadratic program
- Other examples

↳ Standard example: Markowitz model.