SOLUTIONS TO HOMEWORK #3, MATH 441, FALL 2018

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Please note:

(1) You may work together on homework, but you must write your own code/software; you must write up your own solutions individually.
(2) You must acknowledge with whom you worked. You must also acknowledge any sources you have used beyond the textbook and class material.
(3) In all these problems you must **justify your answer** unless the problem states otherwise; you will not be given any credit for stating the correct answer without a **written justification** that your answer is correct.
(4) Submit the entire homework as a single PDF file to `canvas.ubc.ca`.

Homework Problems

Class presentations will take place during the last three weeks of November, during class; each day will have at most three time slots, and each group must present once. (So the first possible day is Nov 12, the second is Nov 14, etc., and the ninth day is Nov 30.) Each group will submit a preference, and we will form a utility function as described in class: for example, if group number 12 expresses the preferences 456123987, then

(1) day 4 (Nov 19) is their first choice, day 5 (Nov 21) their second choice, and day 7 (Nov 26) is their last choice, and
(2) we add the utility term of Group 12

$$9x_{12,4} + 8x_{12,5} + 7x_{12,6} + 6x_{12,1} + 5x_{12,2} + 4x_{12,3} + 3x_{12,9} + 2x_{12,8} + x_{12,7}$$

to the utility function.

(In general $x_{i,j}$ is the utility of day j to Group i, based on the declared preferences of Group i.) Assume that there are 18 groups and answer the following questions.

(1) Say that each group has the same preferences, say 123456789. Can each group get its first choice in an optimal schedule? Could some group get its last choice in an optimal schedule? Is there a unique optimal schedule in this case?

Research supported in part by an NSERC grant.

Solution: Not all groups can get their first choice, since they all prefer the same day and there are only three slots. No group will be assigned day 9 in an optimum utility solution: in any schedule that assigns a group—say Group i—to day 9, there is an empty slot in days 1–8 (which can accomodate 24 groups), and hence one gets a higher utility by scheduling Group i into one of those empty slots. There is no unique optimal solution: any optimum utility solution must leave days 7,8,9 free (reasoning similarly to the last sentence); hence **any** assignment of the groups to days 1–6 is a feasible and optimum solution.

(2) Under any set of preferences, can a group be assigned its last choice in an optimal schedule?

Solution: No: in any feasible solution, if Group i is assigned its last choice, then there is an empty slot in the eight days it prefers to its last choice (there are 24 slots in these eight days, and only 17 other groups); hence we can fix the assignment of the 17 other groups and move Group i to a slot in one of these eight days, which gives another feasible solution with a higher objective value.

(3) Under any set of preferences and in any optimal schedule, is it possible that Groups 1 and 2 would rather switch days? (In other words, Group 1 prefers the day assigned to 2 than the day assigned to them, and Group 2 prefers the day assigned to Group 1 than the day assigned to them.)

Solution: No: in any feasible solution, if Groups 1 and 2 would rather switch days, then by switching them we get another feasible solution where the contribution of the objective function for Groups 1 and 2 is larger, and that of the other groups remains the same. Hence a feasible solution cannot be optimal if Groups 1 and 2 would rather switch days.

(4) Say that we multiply the utility term of Group 1 by a real number α with $\alpha > 1000$ (and don't change the other groups' utility terms). Describe how this changes the optimal solution.

Solution: We claim that Group 1 gets its first choice, and that the other 17 groups are schedule: in any feasible solution, say that Group 1 does not get its first choice, which is day j. If day j is not full, then we can move Group 1 to day j and (maintain feasibility and) increase the objective. Then if day j is full, we can swap Group 1 and some Group i that occupies day j; the objective is increased by at least α in Group 1's contribution, at loses at most $9 - 1 = 8$ for Group i's loss. Hence Group 1 always gets its first choice; the rest of algorithm assigns the remaining groups to the remain slots to maximize the objective function as before. [Note: This argument works for any $\alpha > 8$; in fact, an argument above shows that Group i will be assigned one of its first

6 choices, hence the loss in the Group i part of the objective function is at most $9 - 4 = 5$; hence the same conclusion holds for $\alpha > 5$.]

(5) Same question for $0 < \alpha < 1/1000$.

Solution: Groups 2–18 are scheduled optimally, ignoring Group 1 (and its contribution to the objective function), and then Group 1 gets its first choice from what is left: indeed, in any optimal solution, if Groups 2–18 are not scheduled optimally when Group 1 is ignored, then we can reschedule Groups 2–18 optimally and then put Group 1 anywhere: the result increases the objective by at least 1 for the contribution of Groups 2–18, and decreases the contribution of Group 1 by at most $(9-1)\alpha = 8\alpha < 0.008$; then the objective increases by at least $1 - 0.008$, which is impossible for an optimal solution. Once Groups 2–18 are scheduled optimally, Group 1 chooses its first choice in what is left (since otherwise the solution is not optimal).

(6) Same question for $\alpha = -1$.

Solution: The upshot is that this is the same optimization problem as before, except the preferences for Group 1 are reversed. There are many ways to see this; here is one way: if Group 1 has preferences $i_1, \ldots, i_9$, then the contribution to the objective function for Group 1 is

$$\begin{aligned} \alpha(9x_{1,i_1} + 8x_{1,i_1} + \cdots + x_{1,i_9}) &= -9x_{1,i_1} - \cdots - x_{1,i_9} \\ &= (1-10)x_{1,i_1} + (2-10)x_{1,i_2} + \cdots + (9-10)x_{1,i_9} \\ x_{1,i_1} + 2x_{1,i_2} + \cdots &+ 9x_{1,i_9} - 10(x_{1,i_1} + \cdots + x_{1,i_9}) \\ &= x_{1,i_1} + 2x_{1,i_2} + \cdots + 9x_{1,i_9} - 10 \end{aligned}$$

since $x_{1,i_1} + \cdots + x_{1,i_9}$ must equal 1. Hence the affect of setting $\alpha = -1$ is the same as subtracting 10 from the objective function (which does not change which solution are optimal) and reversing the preferences of Group 1.

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