SOLUTIONS TO HOMEWORK #2, MATH 441, FALL 2018

JOEL FRIEDMAN

Please note:

(1) You may work together on homework, but you must write your own code/software; you must write up your own solutions individually.
(2) You must acknowledge with whom you worked. You must also acknowledge any sources you have used beyond the textbook and class material.
(3) In all these problems you must **justify your answer** unless the problem states otherwise; you will not be given any credit for stating the correct answer without a **written justification** that your answer is correct.
(4) Submit the entire homework as a single PDF file to `canvas.ubc.ca`.

Homework Problems

An $n \times n$ *Latin square* is an $n \times n$ array labeled with the numbers $\{1, 2, \ldots, n\}$ such that each row contains one of each number and one column contains one of each number. Here are some examples:

$$\begin{matrix} 1 & 2 \\ 2 & 1 \end{matrix} \qquad \begin{matrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{matrix} \qquad \begin{matrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{matrix}$$

A *Latin square puzzle* is a partially filled in Latin square, array, and the "puzzle" is to complete it to get a Latin square, for example:

$$\begin{matrix} ? & ? \\ 2 & ? \end{matrix} \qquad \begin{matrix} 1 & 2 & ? \\ ? & ? & 1 \\ ? & ? & ? \end{matrix} \qquad \begin{matrix} 1 & ? & ? \\ ? & 1 & ? \\ ? & ? & 2 \end{matrix}$$

(there can be no way to do this, or many ways).

(1) Solve the first 2×2 magic square problem using IP (integer programming) and some software that solves IP's (e.g., Gurobi) as follows:

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(a) introduce 8 decision variables $x11c1, x11c2, x12c1, \ldots, x22c2$, where

$$x12c1 = \begin{cases} 1 & \text{if the square in row 1 and column 2 is labeled 1, and} \\ 0 & \text{otherwise,} \end{cases}$$

and similarly for the other 7 variables.

(b) Write out equations for the following constraints:

(i) each square in the array has exactly one label; for example

$$x11c1 + x11c2 = 1$$

expresses the fact that the square in row 1 and column 1 has exactly one colour;

(ii) each row has one of each label; for example

$$x11c1 + x12c1 = 1$$

expresses the fact that row 1 has colour 1 occurring exactly once;

(iii) each column has one of each label;

(iv) the decision variables take values in $\{0, 1\}$, or equivalently (given the constraints) they take values that are integers and non-negative;

(v) the particular puzzle tells you that $x21c2 = 1$, i.e., the square in the second row and first column is labeled 2. [Since an earlier version of the homework had a typo, it is OK if you solve the condition $x21c1 = 1$, i.e., the square in the second row and first column is labeled 1.]

(c) the objective function is unimportant (but some objective may need to be specified, depending on your software package).

Solution: See the Gurobi *.lp file and the text file of the session; Gurobi found the solution $x11c1, x21c2, x12c2, x22c1 = 1$, which corresponds to the Latin square:

$$\begin{matrix} 1 & 2 \\ 2 & 1 \end{matrix}$$

(2) (a) If you solve the $n \times n$ Latin square puzzle as an IP, in the analogous manner as above, how many constraints would you write down (as a function of n, excluding given labels on specific squares particular to a puzzle)? **Justify your answer; you will get no credit if you simply give the function of n.**

Solution: Each of n^2 squares must have one colour, which gives n^2 constraints. Each row has a constraint for each colour; since there are n rows and n colours, we have n^2 row constraints; similarly we have n^2 column constraints. Hence the total number of constraints is $3n^2$.

(b) An equivalent way to describe the $n \times n$ Latin square problem is as a graph colouring problem, where certain pairs of squares are not allowed to have the same label. As an example, for $n = 3$, the fact that squares $(1, 1)$ and $(1, 2)$ don't have the same colour amounts to three inequalities

$$x11c1 + x12c1 \leq 1, \quad x11c2 + x12c2 \leq 1, \quad x11c3 + x12c3 \leq 1.$$

How many constraints would you write down for this way of formulating the Latin Square problem (as a function of n, excluding given labels on specific squares particular to a puzzle)? **Justify your answer; you will get no credit if you simply give the function of n.**

Solution: Again, each of n^2 squares must have one colour, which gives n^2 constraints. Each row has n squares, and pair of these squares has n constraints (they must be coloured differently); the number of pairs among n elements is

$$(n-1) + (n-2) + \cdots + 2 + 1 = \frac{n(n-1)}{2}.$$

So each row gives rise to $n^2(n-1)/2$ constraints, and since there are n rows this gives $n^3(n-1)/2$ total constraints over all rows. Similarly each column has $n^3(n-1)/2$. So the total number of constraints is

$$n^2 + \frac{n^3(n-1)}{2} + \frac{n^3(n-1)}{2} = n^2 + n^3(n-1) = n^4 - n^3 + n^2.$$

(c) Which of the two formulations of the Latin square IP involves fewer constraints? **Justify your answer.**

Solution: We claim that for $n \geq 3$,

$$n^4 - n^3 + n^2 \geq 3n^2 :$$

to check this we divide by n^2, so it is equivalent to see if

$$n^2 - n + 1 \geq 3,$$

i.e., $n(n-1) \geq 2$; but for $n \geq 3$ we have $n(n-1) \geq 3(3-1) = 6$. (The more serious problem is that $n^4 - n^3 + n^2$ is a quartic in n, which is much larger than $3n^2$ (quadratic in n) for large n; so writing the Latin square problem as a colouring problem becomes impractical for a much smaller value of n than in the first method.)

Department of Computer Science, University of British Columbia, Vancouver, BC V6T 1Z4, CANADA, and Department of Mathematics, University of British Columbia, Vancouver, BC V6T 1Z2, CANADA.

E-mail address: jf@cs.ubc.ca or jf@math.ubc.ca

URL: http://www.math.ubc.ca/~jf