SOLUTIONS TO HOMEWORK #1, MATH 441, FALL 2018

JOEL FRIEDMAN

Please note:

(1) You may work together on homework, but you must write your own code/software; you must write up your own solutions individually.
(2) You must acknowledge with whom you worked. You must also acknowledge any sources you have used beyond the textbook and class material.
(3) In all these problems you must **justify your answer** unless the problem states otherwise; you will not be given any credit for stating the correct answer without a **written justification** that your answer is correct.
(4) Submit the entire homework as a single PDF file to `canvas.ubc.ca`.

Homework Problems.

(1) Say that you allow yourself at most 12 hours of TV per day, which consists of
   (a) x_1 hours of "The Expanse" reruns,
   (b) x_2 hours of "The X-Files" reruns,
   (c) x_3 hours of "The Walking Dead" reruns,
   (d) x_4 hours of other documentary programs,

   which gives you a utility of
$$U(\mathbf{x}) = 10x_1 + 9x_2 + 5x_3 + 2x_4 \ .$$
   In addition, you are allowed to watch at most 2.3 hours of The Expanse, and at most 3.9 hours of The X-Files.
   (a) Write an LP that finds the TV viewing that maximizes your utility given the the above constraints; no justification is needed for this part.

Research supported in part by an NSERC grant.

Solution:

$$\begin{aligned} \max \quad 10x_1 + 9x_2 + 5x_3 + 2x_4 \quad &\text{s.t.} \\ x_1 + x_2 + x_3 + x_4 &\le 12 \\ x_1 &\le 2.3 \\ x_2 &\le 3.9 \\ x_1, \dots, x_4 &\ge 0 \end{aligned}$$

(b) What is the optimal solution of the above optimization problem? **Justify your answer**.

Solution: $x_1^* = 2.3$, $x_2^* = 3.9$, and $x_3 = 5.8$. [There are many ways to justify this; here is one way.]

(i) First we claim that $x_1 + x_2 + x_3 + x_4 = 12$ in an optimal solution, for otherwise $x_1 + x_2 + x_3 + x_4 < 12$ and we could increase x_3 while fixing the other variables, which would increase the objective.

(ii) Next we claim that $x_1 = 2.3$ in optimality, since otherwise $x_1 < 2.3$ and $x_2 + x_3 + x_4 > 9.7$; hence we could increase x_1 some amount in exchange for the same decrease in one of x_2, x_3, x_4, preserving feasibility and increasing the objective (since x_1 has the highest coefficient in the objective).

(iii) Similarly $x_2 = 3.9$ in optimality, for otherwise we could increase the objective by increasing x_2 and decreasing x_3 and/or x_4.

(iv) Similarly $x_3 = 5.8$, for otherwise $x_4 > 0$ and we can increase the objective by putting the x_4 TV hours into x_3.

Hence the optimal solution is $x_1^* = 2.3$, $x_2^* = 3.9$, and $x_3 = 5.8$.

(c) Use Gurobi (or some other program) to solve this LP. Provide a printout of your code/software (encoding this LP), and of the solution Gurobi provides. If you don't use Gurobi, you have to indicate what your code/software is doing.

Solution: See supporting Gurobi txt files on the course homepage. Since the LP in this question is similar to examples done in class on TV watching, I have simply modified one of these .lp files.

(d) What is the optimal solution and objective (do not use scientific notation)?

Solution: $x_1^* = 2.3$, $x_2^* = 3.9$, $x_3^* = 5.8$, $x_4^* = 0$, objective is 87.1 (reported as 8.710000000e+01 in the Gurobi session). See supporting Gurobi txt files.

(e) Use Gurobi or your software to solve the same problem with the additional constraint that $x_1, \dots, x_4$ have to be integers.

Solution: $x_1^* = 2$, $x_2^* = 3$, $x_3^* = 7$, $x_4^* = 0$, objective is 82. See supporting txt files.

(2) Consider the following variant of Exercise 1: $x_1, \ldots, x_4$ are the same and satisfy the same constraints, but your utility function involves a parameter $B \in \mathbb{R}$ and is given as

$$U(\mathbf{x}) = 10x_1 + 9x_2 + 5x_3 + 2x_4 + B(12 - x_1 - x_2 - x_3 - x_4),$$

[which reflects an additional utility of B per each hour of TV that you don't watch]. Describe the optimal solutions for all values of $B \in \mathbb{R}$, and **justify your answer** (explain this in words; your justification can appeal to the simplex method, but this is not recommended or required). [Hint: The optimal solution changes as B passes through certain values, such as $B = 10$: the optimal solution for $B \geq 10$ is different than that for $10 \geq B \geq 9$.]

Solution: If we set $x_5 = 12 - x_1 - x_2 - x_3 - x_4$, then the LP becomes

$$\max \quad 10x_1 + 9x_2 + 5x_3 + 2x_4 + Bx_5 \quad \text{s.t.}$$

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 &\leq 12 \\ x_1 &\leq 2.3 \\ x_2 &\leq 3.9 \\ x_1, \ldots, x_5 &\geq 0 \end{aligned}$$

Arguing as in the first problem, the optimal solutions become:

(a) $B > 10$: $x_5^* = 12$, $x_1^* = x_2^* = x_3^* = x_4^* = 0$, since for $B = 10$ the variable x_5 has the highest objective coefficient (and the $x_1, \ldots, x_5$ are interchangeable, within their bounds).

(b) $9 < B < 10$: Similarly $x_1^* = 2.3$, $x_5^* = 9.7$, $x_2^* = x_3^* = x_4^* = 0$, since x_1 has the highest coefficient, and then x_5; hence we let x_1 to be its maximum value, and put the rest of the hours into x_5.

(c) $5 < B < 9$: Similarly $x_1^* = 2.3$, $x_2^* = 3.9$, $x_5^* = 9.7$, $x_3^* = x_4^* = 0$, since x_5's coefficient is between that of x_2 and x_3.

(d) $B < 5$: Similarly $x_1^* = 2.3$, $x_2^* = 3.9$, $x_3^* = 9.7$, $x_5^* = x_4^* = 0$, since x_5's coefficient is lower than that of x_3.

For cases on the border, such as $B = 10$, $B = 9$, and $B = 5$, there are infinitely many solutions. For example, if $B = 10$ then either the solution for $B > 10$ or $9 < B < 10$ works, but you can have any "convex combination" of these solutions, i.e., and combination of x_1 and x_5 with $x_1 + x_5 = 12$ and $0 \leq x_1 \leq 2.3$.

Department of Computer Science, University of British Columbia, Vancouver, BC V6T 1Z4, CANADA, and Department of Mathematics, University of British Columbia, Vancouver, BC V6T 1Z2, CANADA.

E-mail address: `jf@cs.ubc.ca or jf@math.ubc.ca`

URL: `http://www.math.ubc.ca/~jf`