SUMMARY OF FIRST EXAMPLE

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(1) LP: max $\vec{c}^T\vec{x}$ subject to $A\vec{x} \le \vec{b}$ and $\vec{x} \ge 0$. Where

$$\vec{c} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 8 \\ 5 \\ 8 \end{bmatrix}.$$

(2) We have $\vec{c}, \vec{b}, A$ all have non-negative components or entries. Typical when have resources making products, where each product has a positive effect on the objective function $z = \vec{c}^T\vec{x}$.

(3) Initial dictionary

$$\begin{aligned} x_3 &= 8 - x_1 - 2x_2 \\ x_4 &= 5 - x_1 - x_2 \\ x_5 &= 8 - 2x_1 - x_2 \\ z &= \quad 4x_1 + 5x_2 \end{aligned}$$

Final dictionary

$$\begin{aligned} x_2 &= 3 + x_4 - x_3 \\ x_1 &= 2 - 2x_4 + x_3 \\ x_5 &= 1 + 3x_4 - x_3 \\ z &= 23 - 3x_4 - x_3 \end{aligned}$$

(4) Understanding $x_1 + x_2 \le 5$ as a constraint on coffee (x_1, x_2 are coffee-based drinks; see notes), we guess that $z = 23 - 3x_4 - x_3$ means that if the 5 units of coffee changes a small amount, then the utility changes by 3 (meaning 3 units of utility per unit of coffee) times the coffee change; less coffee means less utility.

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(5) Dual LP is minimize $8y_1 + 5y_2 + 8y_3$ subject to $4 \le y_1 + y_2 + 2y_3$, $5 \le 2y_1 + y_2 + y_3$ and $y_1, y_2, y_3 \ge 0$. Derivation in matrix notation:

$$A\vec{x} \le \vec{b} \implies \vec{y}^T A\vec{x} \le \vec{y}^T \vec{b}$$

provided that $\vec{y} \ge \vec{0}$. Hence if $\vec{y}$ is *dual feasible* in the sense that

$$\vec{c}^T \le \vec{y}^T A, \quad \text{and} \quad \vec{y} \ge \vec{0}, \tag{1}$$

then for any *feasible* $\vec{x}$ (meaning that $A\vec{x} \le \vec{b}$ and $\vec{x} \ge \vec{0}$), we always have

$$z = z(\vec{x}) = \vec{c}^T \vec{x} \le \vec{y}^T A\vec{x} \le \vec{y}^T \vec{b},$$

i.e., the objective $z = z(\vec{x})$ is never larger than $\vec{y}^T \vec{b}$. Hence the dual LP is to minimize $\vec{y}^T \vec{b}$ subject to (1). To write this in standard form, notice that $\vec{c}^T \le \vec{y}^T A$ is equivalent (after taking the transpose of both sides) to $-A^T \vec{y} \le -\vec{c}$; hence the dual LP becomes

$$\text{maximize } -\vec{b}^T \vec{y} \quad \text{subject to}$$
$$-A^T \vec{y} \le -\vec{c}, \quad \vec{y} \ge \vec{0}\ .$$

So in the primal LP, the roles of $A, \vec{c}, \vec{b}$ become, in the dual LP, $-A^T, -\vec{b}, -\vec{c}$.

(6) Concretely, the dual to the LP in our first example (above) is to maximize $-8y_1 - 5y_2 - 8y_3$ subject to $-y_1 - y_2 - 2y_3 \le -4$ and $-2y_1 - y_2 - y_3 \le -5$.

(7) Each primal dictionary has a dual dictionary: for example, the initial dual dictionarly would be

$$\begin{aligned} y_4 &= -4 + y_1 + y_2 + 2y_3 \\ y_5 &= -5 + 2y_1 + y_2 + y_3 \\ w &= \qquad -8y_1 - 5y_2 - 8y_3 \end{aligned}$$

This dictionary is not feasible, but is *dual feasible*. However, the final dictionary of the original LP has the dual dictionary

$$\begin{aligned} y_2 &= 3 - y_5 + 2y_4 - 3y_3 \\ y_1 &= 1 + y_5 - y_4 + y_3 \\ w &= -23 - 3y_5 - 2y_4 - y_3 \end{aligned}$$

which is feasible and *dual feasible* (meaning negative coefficients in the w row), and hence is also optimal.

(8) From this observation, we see that if any LP is feasible and bounded, then—because the simplex method always finds an optimal solution in a final dictionary that is both feasible and dual feasible—the dual LP has a corresponding feasible and dual feasible optimal dictionary, with a matching optimal objective value.

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