HOMEWORK #6, MATH 441, FALL 2017

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(1) Let $\mu > 0$ be a real parameter, and consider the problem of minimizing $f(w_1, w_2)$ subject to $g_i(w_1, w_2) \le 0$ for $i = 1, \ldots, 4$, where

$$f(w_1, w_2) = 100 - 4\mu(w_1^2 - 2w_1w_2 + w_2^2), \quad g_1(w_1, w_2) = w_1 + w_2 - 10$$

$$g_2(w_1, w_2) = -w_1 - w_2 + 10, \quad g_3(w_1, w_2) = -w_1, \quad g_4(w_1, w_2) = -w_2.$$

(Note the similarity to Problem 2 from Homework 4.) Answer the following questions and justify your answer:

(a) Describe the feasible region of this program as a subset of $(w_1, w_2) \in \mathbb{R}^2$.

(b) For each feasible (w_1, w_2), describe which of the $g_i \le 0$ are active constraints. [You may draw a diagram or make a list for each subset of $i = 1, 2, 3, 4$, but you should justify your answer in words either way.]

(c) Find all KKT points of this program.

(d) Relate your findings to the solution of Problem 2 of Homework 4.

Solution:

(a) The feasible region are points of the form $(w_1, 10 - w_1)$ with $0 \le w_1 \le 10$.

(b) g_1 and g_2 are active in the entire region. In addition, g_3 is active at $(0, 10)$, and g_4 is active at $(10, 0)$.

(c) We have

$$\nabla f = -4\mu(2w_1 - 2w_2, -2w_1 + 2w_2) = -8\mu(w_1 - w_2)(1, -1), \quad \nabla g_1 = (1, 1)$$

$$\nabla g_2 = (-1, -1), \quad \nabla g_3 = (-1, 0), \quad \nabla g_4 = (0, -1).$$

The KKT points are therefore as follows:

(i) For $(w_1, 10 - w_1)$ with $0 < w_1 < 10$, i.e., when g_1, g_2 are active, a point is KKT iff there are $u_1, u_2 \ge 0$ such that

$$(0, 0) = \nabla f + u_1 \nabla g_1 + u_2 \nabla g_2 = -8\mu(w_1 - w_2)(1, -1) + (u_1 - u_2)(1, 1).$$

This is equivalent to the two equations

$$0 = -8\mu(w_1 - w_2) + (u_1 - u_2), \quad 0 = -8\mu(w_1 - w_2) - (u_1 - u_2).$$

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Adding the two equations we have $0 = -16\mu(w_1 - w_2)$ so that $w_1 = w_2 = 10 - w_1$, and hence we must have $w_1 = w_2 = 5$. If $w_1 = w_2 = 5$, then the two equations amount to $u_1 = u_2$, for which there are (infinitely many) solutions with $u_1 = u_2 \geq 0$.

Hence for $0 < w_1 < 10$, there is a single KKT point, namely $(w_1, w_2) = (5, 5)$.

(ii) For $(10, 0)$, i.e., when g_1, g_2, g_4 are active, the KKT condition holds if we can find $u_1, u_2, u_4 \geq 0$ for which

$$(0,0) = \nabla f + u_1 \nabla g_1 + u_2 \nabla g_2 + u_4 \nabla g_4 = -80\mu(1,-1) + (u_1 - u_2)(1,1) + u_4(0,-1),$$

which is equivalent to the two equations

$$0 = -80\mu + (u_1 - u_2), \quad 0 = 80\mu + (u_1 - u_2) - u_4\ ;$$

the first equation says that $u_1 - u_2 = 80\mu$, and the second then is equivalent to saying that $u_4 = 80\mu + (u_1 - u_2) = 160\mu$. So $u_4 = 160\mu \geq 0$, and we may take any $u_2 \geq 0$ and set $u_1 = 80\mu + u_2$ to get a solution to the KKT equation with $u_1, u_2, u_4 \geq 0$.

Hence $(10, 0)$ is a KKT point.

(iii) For $(0, 10)$, i.e., when g_1, g_2, g_3 are active, the KKT condition holds if we can find $u_1, u_2, u_3 \geq 0$ for which

$$(0,0) = \nabla f + u_1 \nabla g_1 + u_2 \nabla g_2 + u_3 \nabla g_3 = -80\mu(-1,1) + (u_1 - u_2)(1,1) + u_3(-1,0),$$

which is equivalent to the two equations

$$0 = 80\mu + (u_1 - u_2) - u_3, \quad 0 = -80\mu + (u_1 - u_2)\ ;$$

the second equation says that $u_1 - u_2 = 80\mu$, and the first then is equivalent to saying that $u_3 = 80\mu + (u_1 - u_2) = 160\mu$. So $u_3 = 160\mu \geq 0$, and we may take any $u_2 \geq 0$ and set $u_1 = 80\mu + u_2$ to get a solution to the KKT equation with $u_1, u_2, u_3 \geq 0$.

Hence $(0, 10)$ is a KKT point.

It follows that the KKT points are $(0, 10), (5, 5), (10, 0)$.

(d) Problem 2 of Homework 4 has the same feasibility region. In this problem we are minimizing the Markowitz utility, rather than maximizing it. So in addition to the KKT point $(5, 5)$ (which appears either when you maximize or minimize), the endpoints of the feasibility region, $(0, 10)$ and $(10, 0)$ are KKT points.

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