HOMEWORK #5, MATH 441, FALL 2017

JOEL FRIEDMAN

(1) Consider our process of scheduling group presentations for MATH 441 this term (see the course website, specifically the webpage on the schedule of presentations). Recall that we formed a utility function based on group preferences for the class day of their presentation; each group presents on exactly one day, and at most three groups can present on any one day. Recall that if group Z expressed preferences 456123987 (so that day 4 is their first choice, day 5 their second choice, and day 7 their last choice), we added the term

$$9Z_4 + 8Z_5 + 7Z_6 + 6Z_1 + 5Z_2 + 4Z_3 + 3Z_9 + 2Z_8 + Z_7$$

to the utility function. Then we used Gurobi to find a utility maximizing schedule.

Assume that there are 16 groups and 9 days over which we schedule them, and that each group has expressed their preferences as above. **Justify your answers** (a simple "yes" or "no" will give you no credit) to the questions below regarding any possible optimal schedule, i.e., utility maximizing schedule, that Gurobi finds.

(a) Say that each group has the same preferences, say 123456789. Can each group get its first choice in an optimal schedule? Could some group get its last choice in an optimal schedule? Is there a unique optimal schedule in this case?

(b) Under any set of preferences, can a group be assigned its last choice in an optimal schedule?

(c) Under any set of preferences and in any optimal schedule, is it possible that groups A and B would rather switch days? (In other words, group A prefers the day assigned to B than the day assigned to them, and group B prefers the day assigned to group A than the day assigned to them.)

(d) Under any set of preferences and in any optimal schedule, is it possible that group A would rather switch days with group B but that group B does not want to switch with group A? If your answer is "yes," give an example of a set of preferences and an optimal solution where this would happen.

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(e) Under any set of preferences and in any optimal schedule, is it possible that group A would rather switch their day to some day that is not fully booked?

(f) Under any set of preferences and in any optimal schedule, is it possible that group A would rather have group B's day, and group B would rather have group C's day, and group C would rather have group A's day?

Solution:

(a) (i) If each group has preferences 123456789, then all groups have day 1 as their top choice, and only three groups can be assigned to day 1; hence not all groups can get their first choice.

(ii) A group will not be assigned day 9 in an optimum utility solution: in any schedule that assigns a group—say group A—to day 9, there is an empty slot in days 1–8 (which can accomodate 24 groups), and hence one gets a higher utility by scheduling group A into one of those empty slots.

(iii) Similarly, any maximum utility schedule must assigns three groups each to days 1,2,3,4,5, for otherwise there would be an empty slot somewhere in these days, and we could put any group assigned to days 6,7,8,9 in this slot an obtain a higher utility. Similarly again, the one group that is not scheduled in days 1–5 must be scheduled in day 6, for otherwise we could schedule this group in day 6. Conversely, any scheduling filling days 1–5 and placing the remaining group in day 6 has the same (optimal) utility $3 \cdot (9+8+7+6+5) + 1 \cdot 4$. Hence the optimal schedule is not unique.

An analogous argument applies to any situation where the 16 groups have the same preferences.

(b) A group, say group A, can never be assigned its last choice: in any schedule there must be at least one empty slot in group A's top 6 preferred days; in any schedule where group A is not assigned to one of its 6 top preferences, one could reassign group A to such an empty slot (without changing the schedule for the other groups) to get a higher utility.

(c) Consider a schedule where A and B would rather switch days, and consider the new schedule where we swap A and B and leave all other groups as they are. This new schedule has the same utility for the other groups, but has a strictly higher utility both for groups A and B. Hence the new schedule has a higher utility. Hence an optimal schedule can never have groups A and B—or any other pair of groups—each wanting to switch days with the other.

(d) If all groups have the same preferences, an optimal schedule may have groups B,C,D in day 1 (see the solution to part (a) above).

Then group A would want to switch with group B, but group B wouldn't want to switch with group A.

(e) In any schedule where group A is assigned a day of lower preference to a day that is not fully booked, one can reassign A to this higher preference day and leave all other groups alone in the schedule. This new schedule would have a higher utility, and hence the old schedule cannot be optimal.

(f) Similarly, if switching group A to group B's slot, B to C's slot, and C to A's slot gives each of groups A,B,C a higher individual utility, then such a switch—leaving all other groups alone—results in a higher utility schedule. Hence such a switching situation cannot exist in an optimal schedule.

(2) Which statement best characterizes the utility function in Problem 1:
 (a) The utility function is a precise measurement of the overall benefit to society in measurable form; for example, people will be nine times wealthier if each group is assigned its first choice (if possible) than if each group is assigned its last choice (if possible).
 (b) The utility function is used because its optimal solutions have certain desirable properties.

Justify your answer in 25 words or fewer.

Solution: Optimal solutions have many desirable properties (given above). However, if a group has negligible preferences, their assigned day is unimportant.

[There are many possible answers, but ideally an answer would both support the second answer and refute the first.]

(3) Which statement best characterizes the Markowitz utility function:
 (a) The utility function is a precise measurement of the overall benefit to society in some measurable form.
 (b) The utility function is used because its optimal solutions have certain desirable properties.

Justify your answer succinctly.

Solution: Optimal solutions have desirable properties such as (1) maximizing the return for any solution with the same variance, (2) minimizing variance for any return with the same return, (3) preferring a mixture of uncorrelated (or negatively correlated) instruments, corresponding to a "diverse portfolio."

For any fixed risk intolerance (i.e., fixed μ in $\overline{R} - \mu\,\mathrm{Var}(R)$), one can construct a lottery ticket (as done in class) that cannot lose money but has a negative utility. In other words, we would have a higher Markowitz utility by tossing the ticket into the garbage. Hence it seems dubious to claim that the Markowitz utility measures a precise benefit to an investor.

Department of Computer Science, University of British Columbia, Vancouver, BC V6T 1Z4, CANADA, and Department of Mathematics, University of British Columbia, Vancouver, BC V6T 1Z2, CANADA.

E-mail address: jf@cs.ubc.ca or jf@math.ubc.ca

URL: http://www.math.ubc.ca/~jf