HOMEWORK #4, MATH 441, FALL 2017

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Please note:

(1) You may work together on homework, but you must write up your own solutions individually. In particular, you must write your own code, spreadsheets, etc.
(2) You must acknowledge with whom you worked (specify their `gradescope.com` email addresses). You must also acknowledge any sources you have used beyond the textbook and class material.
(3) When you submit your homework to `gradescope.com`, you need to put the solutions to different problems on different pages; `gradescope.com` will ask you to identify which pages correspond to which problems.

(1) Consider the solution (**??**) to the Markowitz utility of the model $R = w_1 A + w_2 M$ subject to $w_1, w_2 \geq 0$ and $w_1 + w_2 = 10$.
  (a) What is w_1^* for the values $\mu = .05$, $\mu = 1$, and $\mu = 1000$?
  (b) Explain intuitively—in terms of μ representing risk aversion—why w_1 is very close to 0 for $\mu = 1000$.
  (c) For the analogous solution (**??**) for the model $R = w_1 A + w_2 B$ (subject to the same conditions), what is the value of w_1^* for the value $\mu = .05$?
  (d) Explain intuitively—in terms of the difference between the models $w_1 A + w_2 M$ and $w_1 A + w_2 B$—why for $\mu = .05$ the value of w_1 is 10 for one of the models and less than 10 for the other.

Solution:
(a) Respectively, $w_1^* = 10, 5/4, 5/4000$.
(b) With $\mu = 1000$ we are very averse to risk and hence want to invest mostly in M (which has zero variance and is therefore "without risk").
(c) $w_1^* = (1/8)(0.05)^{-1} = 1/(.4) = 2.5$.
(d) The return on the riskless instruments M, B are respectively $0, 9$, and hence higher for B. Hence, for the same amount of risk aversion, we will spend more on B in a mix of A and B than in

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M in a mix of A and M. Hence it makes sense that for certain values of μ we may invest everything in A and nothing in M in the optimal investment, and when we replace M by B it becomes more desirable to invest something in B.

(2) Compute the Markowitz Utility $U(w_1, w_2, \mu)$ for the portfolio $R = w_1 A + w_2 N$. [Note the formula $\mathrm{Corr}(A, N) = -1$ in the Section 7, and note that the formulas in Section 8 imply that

$$\mathrm{Var}(w_1 X + w_2 Y) = w_1^2 \,\mathrm{Var}(X) + 2w_1 w_2 \,\mathrm{Cov}(X, Y) + w_2^2 \,\mathrm{Var}(Y)\ .]$$

Then find the optimum feasible solution for this model under the conditions $w_1, w_2 \geq 0$ and $w_1 + w_2 = 10$.

Solution: We have

$$\mathrm{Cov}(A, N) = \mathrm{Corr}(A, N)\sqrt{\mathrm{Var}(A)\,\mathrm{Var}(N)} = -4,$$

and hence

$$\begin{aligned}\mathrm{Var}(w_1 A + w_2 N) &= w_1^2 \,\mathrm{Var}(A) + 2w_1 w_2 \,\mathrm{Cov}(A, N) + w_2^2 \,\mathrm{Var}(N) \\ &= w_1^2 4 - 2w_1 w_2 4 + w_2^2 4.\end{aligned}$$

Under the condition $w_2 = 10 - w_1$ we therefore have

$$\mathrm{Var}(w_1 A + w_2 N) = w_1^2 4 - 2w_1(10 - w_1)4 + (10 - w_1)^2 4 = 16w_1^2 - 160w_1 + 400.$$

We also have

$$\overline{w_1 A + w_2 N} = 10w_1 + 10w_2 = 10w_1 + 10(10 - w_1) = 100.$$

Hence the Markowitz utility is

$$\begin{aligned}U(\mu; w_1 A + w_2 N) &= \overline{w_1 A + w_2 N} - \mu \,\mathrm{Var}(w_1 A + w_2 N) \\ &= 100 - \mu\big(16w_1^2 - 160w_1 + 400\big).\end{aligned}$$

Differentiating in w_1 we see that the maximum is attained when $32w_1 - 160 = 0$ or $w_1 = 5$.

[This should make sense, since A, N have the same expected return, so $\overline{w_1 A + w_2 N} = 100$ regardless of the investment, and $A + N = 20$ is "riskless," and so $5A + 5N = 100$ is also "riskless."]

(3) Compute the Markowitz Utility $U(w_1, w_2, \mu)$ for the portfolio $R = w_1 C + w_2 P$. Then find **all** optimum feasible solutions for this model under the conditions $w_1, w_2 \geq 0$ and $w_1 + w_2 = 10$.

Solution: First solution: since $C = P$, if $w_1 + w_2 = 10$, then $w_1 C + w_2 P = w_1 C + w_2 C = 10C$. Hence any w_1^* with $0 \leq w_1^* \leq 10$ gives the same (and therefore optimal) utility.

Second solution: We can also find this via computation: under the condition $w_2 = 10 - w_1$ we have:

$$\overline{w_1 C + w_2 P} = w_1 10 + w_2 10 = w_1 10 + (10 - w_1)10 = 100,$$

and $\mathrm{Cov}(C, P) = \mathrm{Cov}(C, C) = 4$ (since $P = C$, or we can derive this using the fact that $\mathrm{Corr}(P, C) = 1$ since they are the same instrument). Hence

$$\mathrm{Var}(w_1C + w_2P) = w_1^2 4 + 2w_1w_2 4 + w_2^2 4$$
$$= w_1^2 4 + 2w_1(10 - w_1)4 + (10 - w_1)^2 4 = 400$$

and hence the Markowitz utility is $100 - \mu 400$, which is independent of w_1. Hence any w_1^* with $0 \leq w_1^* \leq 10$ gives the optimal utility.

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