HOMEWORK #2, MATH 441, FALL 2017

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Please note:

(1) You may work together on homework, but you must write up your own solutions individually. In particular, you must write your own code, spreadsheets, etc.

(2) You must acknowledge with whom you worked (specify their `gradescope.com` email addresses). You must also acknowledge any sources you have used beyond the textbook and class material.

(3) When you submit your homework to `gradescope.com`, you need to put the solutions to different problems on different pages; `gradescope.com` will ask you to identify which pages correspond to which problems.

(1) Use branch and bound to solve the integer linear program max $\vec{c}^T\vec{x}$ subject to $A\vec{x} \le \vec{b}$ and $\vec{x} \ge 0$ and $\vec{x} \in \mathbb{Z}^2$ (i.e., x_1, x_2 must be integers)

$$\vec{c} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 3 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 29.8 \\ 7.3 \\ 8.2 \end{bmatrix}.$$

Do not make use of the specific properties of $A, \vec{b}, \vec{c}$ in this problem (i.e., that they all have non-negative entries/coefficients). Specifically:

(a) Enter the corresponding LP into your LP software; you should find that the optimum solution is $x_1 = 0.9$, $x_2 = 6.4$, and $z = 22.8$.

(b) Try the following branches: $x_2 \le 6$ and $x_2 \ge 7$. If you need to explore the $x_2 \le 6$ branch further, divide this branch into $x_2 \le 5$ and $x_2 = 6$; if you need to explore the $x_2 \ge 7$ branch further, divide this branch into $x_2 = 7$ and $x_2 \ge 8$. (You should find that the branch $x_2 \ge 8$ is infeasible.)

(c) When you reach a branch with x_2 fixed, branch on x_1 in a similar fashion (solve the relaxed LP, and round up and down).

(d) Complete the branch and bound, and make a diagram of the result.

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Solution: First LP solution: $(\mathbf{x_1}, \mathbf{x_2}, \mathbf{z}) = (\mathbf{0.9}, \mathbf{6.4}, \mathbf{22.8})$. Branching according to the instructions on the homework, the branch tree that we will explore looks like this:

(a) Branch 1: $x_2 \leq 6$.
  (i) Sub-branch 1.1: $x_2 = 6$.
  (ii) Sub-branch 1.2: $x_2 \leq 5$.
(b) Branch 2: $x_2 \geq 7$.
  (i) Sub-branch 2.1: $x_2 \geq 8$.
  (ii) Sub-branch 2.2: $x_2 = 7$.

Now we have to start exploring the sub-branches. Since we want to quickly find a feasible integral solution, we should explore sub-branch 1.1 ($x_2 = 6$) or 2.2: ($x_2 = 7$).

Let's begin exploring sub-branch 2.2: we solve the LP with the constraint $x_2 = 7$; the optimal solution turns out to be $(\mathbf{x_1}, \mathbf{x_2}, \mathbf{z}) = (\mathbf{0.3}, \mathbf{7}, \mathbf{22.2})$. Hence we have two further sub-branches to explore

(a) Sub-sub-branch 2.2.1: $x_1 \geq 1$.
(b) Sub-sub-branch 2.2.2: $x_1 \leq 0$.

Solving both these LPs (since will to in order to fully explore this branch) gives the optimal solutions:

(a) Sub-sub-branch 2.2.1: $x_1 \geq 1$. **Infeasible.**
(b) Sub-sub-branch 2.2.2: $x_1 \leq 0$. $(\mathbf{x_1}, \mathbf{x_2}, \mathbf{z}) = (\mathbf{0}, \mathbf{7}, \mathbf{21})$ (which is an **integral feasible solution!**).

So far our tree looks like the following:

Root: $(\mathbf{x_1}, \mathbf{x_2}, \mathbf{z}) = (\mathbf{0.9}, \mathbf{6.4}, \mathbf{22.8})$
(a) Branch 1: $x_2 \leq 6$. **Not yet explored**
  (i) Sub-branch 1.1: $x_2 = 6$. **Not yet explored**
  (ii) Sub-branch 1.2: $x_2 \leq 5$. **Not yet explored**
(b) Branch 2: $x_2 \geq 7$. $(\mathbf{x_1}, \mathbf{x_2}, \mathbf{z}) = (\mathbf{0.3}, \mathbf{7}, \mathbf{22.2})$
  (i) Sub-branch 2.1: $x_2 \geq 8$. **Not yet explored**
  (ii) Sub-branch 2.2: $x_2 = 7$. $(\mathbf{x_1}, \mathbf{x_2}, \mathbf{z}) = (\mathbf{0.3}, \mathbf{7}, \mathbf{22.2})$
    (A) Sub-sub-branch 2.2.1: $x_1 \geq 1$. **Infeasible.**
    (B) Sub-sub-branch 2.2.2: $x_1 \leq 0$. $(\mathbf{x_1}, \mathbf{x_2}, \mathbf{z}) = (\mathbf{0}, \mathbf{7}, \mathbf{21})$

And the best feasible integral solution that we have found has $\mathbf{z} = \mathbf{21}$.

Now we try to eliminate some other branches. Solving Sub-branch 2.1 ($x_2 \geq 8$) yields: infeasible, which completes all of branch 2.

So all that is left is to descend and search branch 1. We solve $x_2 \leq 6$ and get the solution $(\mathbf{x_1}, \mathbf{x_2}, \mathbf{z}) = (\mathbf{1.1}, \mathbf{6}, \mathbf{22.4})$. Since the z value here is greater than 21, which is our current best, we have to descend the tree further. We solve both LP's on sub-branches 1.1 and 1.2 which gives us:

Root: $(\mathbf{x_1}, \mathbf{x_2}, \mathbf{z}) = (\mathbf{0.9}, \mathbf{6.4}, \mathbf{22.8})$
(a) Branch 1: $x_2 \leq 6$. $(\mathbf{x_1}, \mathbf{x_2}, \mathbf{z}) = (\mathbf{1.1}, \mathbf{6}, \mathbf{22.4})$
  (i) Sub-branch 1.1: $x_2 = 6$. $(\mathbf{x_1}, \mathbf{x_2}, \mathbf{z}) = (\mathbf{1.1}, \mathbf{6}, \mathbf{22.4})$
  (ii) Sub-branch 1.2: $x_2 \leq 5$. $(\mathbf{x_1}, \mathbf{x_2}, \mathbf{z}) = (\mathbf{1.6}, \mathbf{5}, \mathbf{21.4})$
(b) Branch 2: $x_2 \geq 7$. $(\mathbf{x_1}, \mathbf{x_2}, \mathbf{z}) = (\mathbf{0.3}, \mathbf{7}, \mathbf{22.2})$

(i) Sub-branch 2.1: $x_2 \geq 8$. **Not yet explored**
(ii) Sub-branch 2.2: $x_2 = 7$. $(\mathbf{x_1}, \mathbf{x_2}, \mathbf{z}) = (\mathbf{0.3}, \mathbf{7}, \mathbf{22.2})$
(A) Sub-sub-branch 2.2.1: $x_1 \geq 1$. **Infeasible.**
(B) Sub-sub-branch 2.2.2: $x_1 \leq 0$. $(\mathbf{x_1}, \mathbf{x_2}, \mathbf{z}) = (\mathbf{0}, \mathbf{7}, \mathbf{21})$

Still nothing in branch 1, i.e., sub-branches 1.1 or 1.2, is eliminated since our best current feasible solution is 21 (from sub-sub-branch 2.2.2). Let's now descend sub-branch $x_2 = 6$, bounding on $x_1 \leq 1$ or $x_2 \geq 2$:

(a) Sub-sub-branch 1.1.1: $x_2 = 6$, $x_1 \geq 2$: **Infeasible**
(b) Sub-sub-branch 1.1.2: $x_2 = 6$, $x_1 \leq 1$: $(\mathbf{x_1}, \mathbf{x_2}, \mathbf{z}) = (\mathbf{1}, \mathbf{6}, \mathbf{22})$, which is a **feasible integral solution** with higher objective value than the best so far.

The new value $z = 22$ is the best integral feasible solution so far. This solution is better than $x_1 \leq 5$, so we eliminate this sub-branch. This gives the complete branch and bound tree below:

Root: $(\mathbf{x_1}, \mathbf{x_2}, \mathbf{z}) = (\mathbf{0.9}, \mathbf{6.4}, \mathbf{22.8})$

(a) Branch 1: $x_2 \leq 6$. $(\mathbf{x_1}, \mathbf{x_2}, \mathbf{z}) = (\mathbf{1.1}, \mathbf{6}, \mathbf{22.4})$
  (i) Sub-branch 1.1: $x_2 = 6$. $(\mathbf{x_1}, \mathbf{x_2}, \mathbf{z}) = (\mathbf{1.1}, \mathbf{6}, \mathbf{22.4})$
    (A) Sub-sub-branch 1.1.1: $x_2 = 6$, $x_1 \geq 2$: **Infeasible**
    (B) Sub-sub-branch 1.1.2: $x_2 = 6$, $x_1 \leq 1$: $(\mathbf{x_1}, \mathbf{x_2}, \mathbf{z}) = (\mathbf{1}, \mathbf{6}, \mathbf{22})$.
  (ii) Sub-branch 1.2: $x_2 \leq 5$. $(\mathbf{x_1}, \mathbf{x_2}, \mathbf{z}) = (\mathbf{1.6}, \mathbf{5}, \mathbf{21.4})$ which was **eliminated** because of $z = 22$ in sub-sub-branch 1.1.2.
(b) Branch 2: $x_2 \geq 7$. $(\mathbf{x_1}, \mathbf{x_2}, \mathbf{z}) = (\mathbf{0.3}, \mathbf{7}, \mathbf{22.2})$
  (i) Sub-branch 2.1: $x_2 \geq 8$. **Infeasible**
  (ii) Sub-branch 2.2: $x_2 = 7$. $(\mathbf{x_1}, \mathbf{x_2}, \mathbf{z}) = (\mathbf{0.3}, \mathbf{7}, \mathbf{22.2})$
    (A) Sub-sub-branch 2.2.1: $x_1 \geq 1$. **Infeasible.**
    (B) Sub-sub-branch 2.2.2: $x_1 \leq 0$. $(\mathbf{x_1}, \mathbf{x_2}, \mathbf{z}) = (\mathbf{0}, \mathbf{7}, \mathbf{21})$

(2) Try the above branch and bound method on the integer program with

$$\vec{c} = \begin{bmatrix} 1 \\ 500 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 100 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 2030 \end{bmatrix}.$$

Specifically, try branch and bound by searching the possible values of x_2 based on the LP relaxation, branching on x_1 values on branches where the x_2 value has been fixed. Then do the same where you first branch on x_1 values, then x_2 values. Is there a significant difference? Explain.

Solution: The original LP has optimal solution $(\mathbf{x_1}, \mathbf{x_2}, \mathbf{z}) = (\mathbf{0}, \mathbf{20.3}, \mathbf{10150})$ Say we branch on $x_2 \leq 20$ and $x_2 \geq 21$. We get

(a) Branch 1: $x_2 \leq 20$. Optimal solution is $(\mathbf{x_1}, \mathbf{x_2}, \mathbf{z}) = (\mathbf{30}, \mathbf{20}, \mathbf{10030})$, which is integral.
(b) Branch 2: $x_2 \geq 21$. **Infeasible.**

Now say we branch on x_1 in the way described in the exercse. We get

(a) Branch 1: $x_1 = 0$. Optimal: $(\mathbf{x_1}, \mathbf{x_2}, \mathbf{z}) = (\mathbf{0}, \mathbf{20}, \mathbf{10000})$.
(b) Branch 2: $x_1 \geq 1$. Optimal: $(\mathbf{x_1}, \mathbf{x_2}, \mathbf{z}) = (\mathbf{1}, \mathbf{20.29}, \mathbf{10146})$.
    (i) Sub-branch 2.1: $x_1 = 1$. Optimal $(\mathbf{x_1}, \mathbf{x_2}, \mathbf{z}) = (\mathbf{1}, \mathbf{20.29}, \mathbf{10146})$.
        (A) Sub-sub-branch 2.1.1: $x_2 \leq 20$. **Not yet explored.**
        (B) Sub-sub-branch 2.1.1: $x_2 \leq 21$. **Not yet explored.**
    (ii) Sub-branch 2.2: $x_1 \geq 2$. Optimal $(\mathbf{x_1}, \mathbf{x_2}, \mathbf{z}) = (\mathbf{2}, \mathbf{20.28}, \mathbf{10146})$.
        (A) Sub-sub-branch 2.2.1: $x_1 = 2$. **Not yet explored.**
        (B) Sub-sub-branch 2.2.2: $x_1 \geq 3$. **Not yet explored.**

We are similarly going to have to expand the x_1 search until we get to $x_1 \geq 30$. Even if we know don't explore sub-sub-branch 2.2.1 $x_1 = 2$, sub-sub-sub-branch 2.2.2.1 $x_1 = 3$, sub-sub-sub-sub-branch 2.2.2.2.1 $x_1 = 4$, etc., and wait for $x_1 \geq 30$, we still solve a lot of LP's that are not very interesting. The problem is that x_1 is much less influential on the objective, z, (given the constraints), and the root LP has $x_1 = 0$, which is far from the optimal $x_1 = 30$ value.

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