Be sure that this examination has 7 pages including this cover

The University of British Columbia

Midterm Examinations - March 2008

Mathematics 340–201

Closed book examination | Time: 50 minutes

Name ______________________ **Signature** ______________________

Student Number ______________________ **Instructor's Name** ______________________

Section Number ______________________

Special Instructions:

Calculators, notes, or other aids may not be used. Answer questions on the exam.

Rules governing examinations

1. Each candidate should be prepared to produce his library/AMS card upon request.

2. Read and observe the following rules:

No candidate shall be permitted to enter the examination room after the expiration of one half hour, or to leave during the first half hour of the examination.

Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.

CAUTION - Candidates guilty of any of the following or similar practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.

(a) Making use of any books, papers or memoranda, other than those authorized by the examiners.

(b) Speaking or communicating with other candidates.

(c) Purposely exposing written papers to the view of other candidates. The plea of accident or forgetfulness shall not be received.

3. Smoking is not permitted during examinations.

1		8
2		5
3		6
4		2
5		6
Total		27

Marks

[8] **1.** Consider the two matrix games

$$A_1 = \begin{bmatrix} 1 & -1 \\ -4 & 4 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & -1 \\ 4 & 4 \end{bmatrix}$$

Assume that A_1 is irreducible (i.e., that every strategy is essential) and use linear algebra to find the value of the game and the equilibrium strategies. How do you know that the irreducibility assumption on A_1 was correct (explain carefully)? Assume that A_2 is irreducible, and try to do the same for A_2; how do you know that the irreducibility assumption on A_2 was wrong (explain carefully)?

$\vec{x}^T \begin{bmatrix} 1 & -1 \\ -4 & 4 \end{bmatrix} = [v \ \ v] \Rightarrow x_1 - 4x_2 = v = -x_1 + 4x_2$ and $x_1 + x_2 = 1$

$\Rightarrow v = 0,\ x_1 = \frac{4}{5},\ x_2 = \frac{1}{5}$. Similarly $\begin{bmatrix} 1 & -1 \\ -4 & 4 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} w \\ w \end{bmatrix}$, $y_1 + y_2 = 1$

$\Rightarrow w = 0,\ y_1 = y_2 = \frac{1}{2}$. So value $= 0$, equ. are $\begin{bmatrix} 4/5 \\ 1/5 \end{bmatrix}$ and $\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$.

Irreducibility assumption correct since $\vec{x}, \vec{y}$ unique and $> \vec{0}$.

$\vec{x}^T \begin{bmatrix} 1 & -1 \\ 4 & 4 \end{bmatrix} = [v \ \ v] \Rightarrow x_1 + 4x_2 = -x_1 + 4x_2 = v$, $x_1 + x_2 = 1 \Rightarrow x_1 = 0, x_2 = 1$, $v = 4$

but $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is not $> \vec{0}$. Also $\begin{bmatrix} 1 & -1 \\ 4 & 4 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} w \\ w \end{bmatrix}$, $y_1 + y_2 = 1$

gives $y_1 - y_2 = 4y_1 + 4y_2 = w$ so $3y_1 + 5y_2 = 0$ gives $\vec{y} = \begin{bmatrix} 5/2 \\ -3/2 \end{bmatrix}$

which is not $> \vec{0}$.

Continued on page 3

[5] **2.** Consider the problem: maximize x_1 subject to $x_1 + x_2 \leq 5$, $x_1 \geq 6$, $x_1, x_2 \geq 0$. Write this as a linear program in standard form. Use the two-phase method, **adding an auxilliary variable x_0 to EVERY slack variable equation in the dictionary,** to show that this linear program is infeasible.

max $z = x_1$ s.t.

$x_1 + x_2 \leq 5$

$-x_1 \leq -6$

$x_1, x_2 \geq 0$

$x_3 = 5 - x_1 - x_2 + x_0$

$x_4 = -6 + x_1 + x_0$

$z = x_1$ $\quad w = -x_0$

x_0 enters, x_4 leaves

$x_0 = 6 - x_1 + x_4$

$x_3 = 11 - 2x_1 - x_2 + x_4$

$w = -6 + x_1 - x_4$

x_1 enters, x_3 leaves

$x_1 = \frac{11}{2} - \frac{1}{2}x_3 - \frac{1}{2}x_2 + \frac{1}{2}x_4$

$x_0 = \frac{1}{2} + \frac{1}{2}x_3 + \frac{1}{2}x_2 + \frac{1}{2}x_4$

$w = -\frac{1}{2} - \frac{1}{2}x_3 - \frac{1}{2}x_2 - \frac{1}{2}x_4$

so w max is $-1/2$

so LP infeasible

Continued on page 4

[6] **3.** Consider the problem: maximize $x_1 + x_2$ subject to $x_1 + 2x_2 \le 4$, $2x_1 + x_2 \le 5$, and $x_1, x_2 \ge 0$. Write the slack variables for this linear program, and write down the dual linear program and dual slack variables.

$x_3 = 4 - x_1 - 2x_2$
$x_4 = 5 - 2x_1 - x_2$

min $4y_1 + 5y_2$ s.t.
$y_1 + 2y_2 \ge 1$
$2y_1 + y_2 \ge 1$
$y_1, y_2 \ge 0$

$y_3 = -1 + y_1 + 2y_2$
$y_4 = -1 + 2y_1 + y_2$

Check to see if the following are optimal solutions to the primal linear program using complementary slackness:

(a) $x_1 = 2$, $x_2 = 1$;

$x_1 = 2$ $y_3 = 0$
$x_2 = 1$ $y_4 = 0$
$x_3 = 0$ $y_1 = ?$
$x_4 = 0$ $y_2 = ?$

$y_1 + 2y_2 = 1$, $2y_1 + y_2 = 1$ $\Rightarrow$ $y_1 = y_2 = \frac{1}{3}$

satisfies comp. slackness, so

$x_1 = 2$, $x_2 = 1$ is optimal

(b) $x_1 = 0$, $x_2 = 2$;

$x_1 = 0$
$x_2 = 2$ $\Rightarrow$ $y_4 = 0$
$x_3 = 0$
$x_4 = 3$ $\Rightarrow$ $y_1 = 0$

$\left.\begin{array}{l} y_3 = -1 + y_1 \\ 0 = -1 + 2y_1 \end{array}\right\} \Rightarrow y_1 = \frac{1}{2},\ y_3 = -\frac{1}{2}$

but $y_3 = -1/2$ is negative, so

$x_1 = 0$, $x_2 = 2$ can't be optimal

Continued on page 5

[2] **4.** Explain why the following linear program must involve a degenerate pivot: maximize x_1 subject to $x_1 \le x_2 + x_3$, $x_1 + x_2 + 4x_3 \le 2$, $x_2 + 5x_3 \le 10$, $3x_1 + 3x_2 + 5x_3 \le 7$, $x_1, x_2, x_3 \ge 0$.

The objective is $z = x_1$, so x_1 enters. $x_1 \le x_2 + x_3$ gives slack variable $x_4 = 0 - x_1 + x_2 + x_3$ which forces x_4 to leave immediately in a degenerate pivot.

[6] **5.** Show that for all $m \times n$ matrix games, A, and stochastic $\mathbf{s}, \mathbf{t}$ of dimension m we have

$$\mathrm{Scream}_{\mathrm{Alice}}((\mathbf{s}+\mathbf{t})/2) \ge (1/2)\mathrm{Scream}_{\mathrm{Alice}}(\mathbf{s}) + (1/2)\mathrm{Scream}_{\mathrm{Alice}}(\mathbf{t})$$

Evaluate each term in this formula for the game Rock–Paper–Scissors, where $\mathbf{s}$ represents "play Rock always" and $\mathbf{t}$ represents "play Scissors always." Does equality hold? [This is standard Rock-Paper-Scissors: rock beats scissors, scissors beats paper, paper beats rock, and each win pays one unit to the winner.]

$$\mathrm{Scream}_A\left(\frac{\vec{s}+\vec{t}}{2}\right) = \mathrm{MinComp}\left(\tfrac{1}{2}\vec{s}^T A + \tfrac{1}{2}\vec{t}^T A\right)$$
$$\ge \mathrm{MinComp}\left(\tfrac{1}{2}\vec{s}^T A\right) + \mathrm{MinComp}\left(\tfrac{1}{2}\vec{t}^T A\right)$$
$$= \tfrac{1}{2}\mathrm{Scream}_A(\vec{s}) + \tfrac{1}{2}\mathrm{Scream}_A(\vec{t}).$$

For "play Rock always", Scream $= -1$ (Betty will play paper)
" " Scissors " ," " " " (" " " rock)

For 50% Rock - 50% Scissors, playing rock gives $-1/2$ payout, which is best for Betty (scissors gives $1/2$, paper gives 0). So
$-1/2 > (1/2)(-1) + (1/2)(-1)$ so equality does NOT hold

Continued on page 6

Continued on page 7

The End

Math 340 Note Sheet for Midterm, Spring 2008

$\text{Scream}_{\text{Alice}}(\mathbf{x}) = \min_{\mathbf{y}\ \text{stoch}} \mathbf{x}^{\mathrm{T}} A\mathbf{y} = \text{MinEntry}(\mathbf{x}^{\mathrm{T}} A)$, $\quad \text{Scream}_{\text{Betty}}(\mathbf{y}) = \max_{\mathbf{x}\ \text{stoch}} \mathbf{x}^{\mathrm{T}} A\mathbf{y} = \text{MaxEntry}(A\mathbf{y})$.

$$\text{value} = \text{MaxScream}_{\text{Alice}} = \text{MinScream}_{\text{Betty}}.$$

If all strategies "essential," (i.e., A irreducible), then are unique solutions to $\mathbf{x}^{\mathrm{T}} A = \mathbf{1}^{\mathrm{T}} v$, $\mathbf{x}^{\mathrm{T}}\mathbf{1} = 1$, $A\mathbf{y} = \mathbf{1}w$, and $\mathbf{y}^{\mathrm{T}}\mathbf{1} = 1$, and they have $v = w$ (the value), $\mathbf{x}, \mathbf{y} > \mathbf{0}$.

$\text{MaxScream}_{\text{Alice}}$ is given by the LP: maximize v subject to $\mathbf{1}^{\mathrm{T}} v \leq \mathbf{x}^T A$, $\mathbf{x}^{\mathrm{T}}\mathbf{1} = 1$, $\mathbf{x} \geq \mathbf{0}$. For example,

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad \text{gives} \quad v \leq x_1 + 3x_2, \quad v \leq 2x_1 + 4x_2, \quad \text{etc.}$$

2-phase method: (1) introduce x_0 on right, (2) pivot x_0 into the basis for a feasible dictionary, and try to minimize $w = -x_0$, (3) if w reaches 0, pivot x_0 out of dictionary and eliminate all x_0; e.g.,

$$\begin{array}{rl} x_4 = & -7 + \cdots + x_0 \\ x_9 = & -8 + \cdots + x_0 \end{array} \quad x_0 \text{ enters, } x_9 \text{ leaves}$$

Degenerate pivots: say x_5 enters, and have $x_3 = 0 + x_2 - 2x_5 + \cdots$ Then x_3 cannot tolerate any positive x_5 value, and leaves without changing the basic feasible solution (and z value).

Bland's rule: to prevent/stop cycling, always break ties in entering/leaving variable by choosing variable with the smallest subscript.

Duality example:

$$\begin{array}{rlcrl} \text{maximize} & 3x_1 + 4x_2 \text{ s.t.} & & \text{minimize} & 7y_1 + 10y_2 \text{ s.t.} \\ 5x_1 + 6x_2 & \leq 7 & & 5y_1 + 8y_2 & \geq 3 \\ 8x_1 + 9x_2 & \leq 10 & & 6y_1 + 9y_2 & \geq 4 \\ x_1, x_2 & \geq 0 & & y_1, y_2 & \geq 0 \end{array}$$

y_1 is what first x inequality is multiplied by to form the dual; so y_1 corresponds to x_3, and similarly y_2 to x_4, and similarly y_3 to x_1 and y_4 to x_2. Dual slacks:

$$y_3 = 3 - 5y_1 - 8y_2, \quad y_4 = 4 - 6y_1 - 9y_2;$$

those must be non-negative.