Marks

[8] **1.** Each of the dictionaries below has x_1 entering variable. Below each dictionary, write which variable leaves (if any); also write "unbounded pivot" (meaning that you know the LP is unbounded at that pivot) and/or "degenerate pivot" if these terms apply. DO NOT ACTUALLY PIVOT.

$$\begin{aligned} x_3 &= 8 - x_1 - 2x_2 \\ x_4 &= 5 - x_1 - x_2 \\ z &= 4x_1 - 5x_2 \end{aligned}$$

Answer: x_4 leaves

$$\begin{aligned} x_3 &= 8 - x_1 - 2x_2 \\ x_4 &= 5 + x_1 - x_2 \\ z &= 4x_1 - 5x_2 \end{aligned}$$

Answer: x_3 leaves

$$\begin{aligned} x_3 &= 8 + x_1 - 2x_2 \\ x_4 &= 5 - x_1 - x_2 \\ z &= 4x_1 - 5x_2 \end{aligned}$$

Answer: x_4 leaves

$$\begin{aligned} x_3 &= 8 + x_1 - 2x_2 \\ x_4 &= 5 + x_1 - x_2 \\ z &= 4x_1 - 5x_2 \end{aligned}$$

Answer: Nothing leaves; unbounded pivot

$$\begin{aligned} x_3 &= 0 + x_1 - 2x_2 \\ x_4 &= 5 - x_1 - x_2 \\ z &= 4x_1 - 5x_2 \end{aligned}$$

Answer: x_4 leaves

$$\begin{aligned} x_3 &= 0 - x_1 - 2x_2 \\ x_4 &= 5 + x_1 - x_2 \\ z &= 4x_1 - 5x_2 \end{aligned}$$

Answer: x_3 leaves; degenerate pivot

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[12] **2.** Consider the two matrix games

$$A_1 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix}$$

For each matrix find (1) the value of the game "Alice Announces," (2) the value of the game "Betty Announces," (3) the duality gap in the "Announce" games, (4) the value and equilibrium strategies of the "Scream" games.

Answer: For A_1, the value of Alice Announces is obtained by taking the minimum in each row, namely 1 for row 1 and 3 for row two, and taking the maximum; hence this value is 3. The value of Betty Announces is the minimum of the maximum values in each row, again 3. So the value of both Announce games are 3, and hence the duality gap is 0. Hence (for 2×2 games) the equilibrium corresponds to the pure strategies employed in the Annouce games, namely Alice chooses strategy (or row) 2, and Betty strategy (or column) 1.

For A_2, the value of Alice Announes is -2, while Betty Announces is 1, for a duality gap of 3. Since the duality gap is positive, we know to solve

$$\vec{x}^{\mathrm{T}} A_2 = v\vec{1}^{\mathrm{T}}, \qquad x_1 + x_2 = 1,$$

to find Alice's equilibrium, $[2/3 \quad 1/3]^{\mathrm{T}}$ with value $v = 0$. Similarly we find Betty's equilibrium via

$$A_2\vec{y} = w\vec{1}, \qquad y_1 + y_2 = 1,$$

yielding $[3/4 \quad 1/4]^{\mathrm{T}}$ (and same value, $w = 0$).

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[6] **3.** Consider the problem: maximize x_1 subject to $x_1 \leq 5$, $x_1 \geq 2$, $x_1 \geq 0$. Write this as a linear program in standard form. Use the two-phase method, **adding an auxilliary variable x_0 to EVERY slack variable equation in the dictionary,** to solve this LP. Use the smallest subscript rule to break any ties for entering or leaving variables.

Answer: Standard form would be: maximize x_1 subject to $x_1 \leq 5$, $-x_1 \leq -2$, $x_1 \geq 0$. We get dictionaries:

$x_2 = 5 - x_1 + x_0$	$x_2 = 7 - 2x_1 + x_3$	$x_2 = 3 - 2x_0 - x_3$
$x_3 = -2 + x_1 + x_0$	$x_0 = 2 - x_1 + x_3$	$x_1 = 2 - x_0 + x_3$
x_0 **enters,** x_3 **leaves**	x_1 **enters,** x_0 **leaves**	

(Note that we have omitted the objective $w = -x_0$ in the above, since it can be read off from the x_0 line.) Now we erase the x_0's, bring in the old objective, z, and continue pivoting:

$x_2 = 3 - x_3$	$x_3 = 3 - x_2$
$x_1 = 2 + x_3$	$x_1 = 5 - x_2$
$z = 2 + x_3$	$z = 5 - x_2$
x_3 **enters,** x_2 **leaves**	**Done!**

This gives the optimum solution $z = x_1 = 5$.

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[8] **4.** Consider our usual LP: maximize $4x_1 + 5x_2$ subject to $x_1 + 2x_2 \leq 8$, $x_1 + x_2 \leq 5$, $2x_1 + x_2 \leq 8$, and $x_1, x_2 \geq 0$. Write the slack variables for this linear program, and write down the dual linear program and dual slack variables.

Answer: See Final 2000, Problem 7, for the slack variables (dictionaries), dual linear program, and correspondences. The slack variables are

$$x_3 = 8 - x_1 - 2x_2, \qquad x_4 = 5 - x_1 - x_2, \qquad x_5 = 8 - 2x_1 - x_2$$

for the primal, and

$$y_4 = -4 + y_1 + y_2 + 2y_3, \qquad y_5 = -5 + 2y_1 + y_2 + y_3$$

for the dual.

Check to see if the following are optimal solutions to the primal linear program using complementary slackness:

(a) $x_1 = 3$, $x_2 = 2$;

Answer: Gives $x_3 = 1$ and $x_4 = x_5 = 0$. The positive x_1, x_2, x_3 force the corresponding y_4, y_5, y_1 (respectively) to be 0; this yields $y_2 = 6$ and $y_3 = -1$. The negative y_3 value tells us that complementary slackness cannot be satisfied for this solution, so it is not optimal.

(b) $x_1 = 2$, $x_2 = 3$;

Answer: (This was done on the 1997 Midterm.) We get $x_3 = x_4 = 0$ and $x_5 = 1$, forcing y_4, y_5, y_3 to be 0; this yields $y_1 = 1$ and $y_2 = 3$. Since all values are non-negative (and satisfy the slack variable equations and complementary slackness), this solution is an optimum solution.

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[4] **5.** Say you took one ϵ (i.e., $\epsilon = \epsilon_1 = \epsilon_2 = \cdots$) instead of m different ϵ's (namely $\epsilon_1 >> \epsilon_2 >> \ldots$) in the perturbation method. Explain, using a formula on the note sheet involving A_B^{-1}, how you could get a degenerate pivot.

Answer: If a variable can enter, and that corresponding entries of $A_B^{-1}\vec{b}$ and $A_B^{-1}\epsilon\vec{1}$ are zero, then we get a degenerate pivot. The condition that $A_B^{-1}\epsilon\vec{1}$ is zero at a given component is equivalent to the corresponding row in A_B^{-1} having a sum of entries equal to zero. (The condition on $A_B^{-1}\vec{b}$ being zero is similarly a linear condition on the corresponding row of $A_B^{-1}\vec{b} = 0$.)

[6] **6.** Consider the game where Alice and Betty each write down a number from 1 to n (where n is a fixed, positive integer), and then reveal it to the other, with the rule that 1 beats 2, 2 beats 3, 3 beats 4, and so on, until $n-1$ beats n, and finally 1 beats n; if their number are not consecutive or n and 1, then the game is a draw. (Here the winner receives one dollar from the loser, and nothing happens in a draw.)

(a) Explain the relationship of Rock/Paper/Scissors to this game when $n = 3$; be specific about how strategies of one game correspond to those of the other.

Answer: There are three correct answers; for example, Rock could correspond to writing down 1, Scissors to 2, and Paper to 3.

(b) Show (as simply as possible) that Alice and Betty playing all strategies $1/n$ of the time give an equilibrium of this game. Be precise about what fact(s) you are using.

Answer: Since every number beats one and loses to one, the "all $1/n$" strategy for Alice gives Betty a value of zero no matter what Betty plays; this gives an Alice Screams value of zero. Similarly that strategy for Betty gives a Betty Screams value of zero. We are now done, using the fact that an Alice Screams value and a Betty Screams value are equal iff this is the value of the game and Alice and Betty's strategies are both equilibrium strategies.

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The End

Be sure that this examination has 8 pages including this cover

The University of British Columbia

Midterm Examinations - March 2009

Mathematics 340–202

Closed book examination — Time: 60 minutes

Name ____________________ **Signature** ____________________

Student Number ____________________ **Instructor's Name** ____________________

Section Number ____________________

Special Instructions:

THIS EXAM IS TWO-SIDED! You will be given a note sheet. Calculators, other notes, or other aids may not be used. Answer questions on the exam.

Rules governing examinations

1		8
2		12
3		6
4		8
5		4
6		6
Total		44