Marks

[8] **1.** Consider the matrix game

$$A = \begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix}$$

For each matrix find (1) the value of the game "Alice Announces," (2) the value of the game "Betty Announces," (3) the duality gap in the "Announce" games, (4) the value and equilibrium strategies of the "Scream" games.

Answer: The value of Alice Announces is obtained by taking the minimum in each row, namely -2 for row one and -3 for row two, and taking the maximum; hence this value is -2. The value of Betty Announces is the minimum of the maximum values in each column, i.e., the minimum of 1 and 4, namely 1. The duality gap is the difference between the announce games, namely $1-(-2)=3$. For the scream game, we therefore know we'll have a mixed strategy. So we solve

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} v & v \end{bmatrix}, \quad x_1 + x_2 = 1$$

to get Alice's equilibrium of $x_1 = 7/10$ and $x_2 = 3/10$, and the game value of $v = -2/10$. Similarly we solve

$$\begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} w \\ w \end{bmatrix}, \quad y_1 + y_2 = 1$$

to get Betty's equilibrium of $y_1 = 6/10$ and $y_2 = 4/10$, and game value (again) of $w = -2/10$.

Continued on page 3

[8] **2.** Consider the problem: maximize $x_1 + x_2$ subject to $x_1 \leq 5$, $x_1 + x_2 \geq 2$, $x_1, x_2 \geq 0$. Write this as a linear program in standard form. Use the two-phase method, **adding an auxilliary variable x_0 to EVERY slack variable equation in the dictionary,** to solve this LP. Use the smallest subscript rule to break any ties for entering or leaving variables.

Answer: Standard form would be: maximize x_1+x_2 subject to $x_1 \leq 5$, $-x_1-x_2 \leq -2$, $x_1, x_2 \geq 0$. We get dictionaries:

$x_3 = 5 - x_1 + x_0$	$x_3 = 7 - 2x_1 - x_2 + x_4$	$x_3 = 3 + 2x_0 + x_2 - x_4$
$x_4 = -2 + x_1 + x_2 + x_0$	$x_0 = 2 - x_1 - x_2 + x_4$	$x_1 = 2 - x_0 - x_2 + x_4$
x_0 **enters,** x_4 **leaves**	x_1 **enters,** x_0 **leaves**	

(Note that we have omitted the objective $w = -x_0$ in the above, since it can be read off from the x_0 line.) Now we erase the x_0's, bring in the old objective, z, and continue pivoting:

$x_3 = 3 + x_2 - x_4$	$x_4 = 3 + x_2 - x_3$
$x_1 = 2 - x_2 + x_4$	$x_1 = 5 - x_3$
$z = 2 + x_4$	$z = 5 + x_2 - x_3$
x_4 **enters,** x_3 **leaves**	**Done—unbounded,**

since x_2 enters but nothing leaves.

Continued on page 4

[8] **3.** Consider our usual LP: maximize $4x_1 + 5x_2$ subject to $x_1 + 2x_2 \leq 8$, $x_1 + x_2 \leq 5$, $2x_1 + x_2 \leq 8$, and $x_1, x_2 \geq 0$. Write the slack variables for this linear program, and write down the dual linear program and dual slack variables.

Answer: See Final 2000, Problem 7, for the slack variables (dictionaries), dual linear program, and correspondences. The slack variables are

$$x_3 = 8 - x_1 - 2x_2, \qquad x_4 = 5 - x_1 - x_2, \qquad x_5 = 8 - 2x_1 - x_2$$

for the primal, and

$$y_4 = -4 + y_1 + y_2 + 2y_3, \qquad y_5 = -5 + 2y_1 + y_2 + y_3$$

for the dual.

Check to see if the following are optimal solutions to the primal linear program using complementary slackness:

(a) $x_1 = 4$, $x_2 = 0$;

Answer: This gives $x_3 = 4$**,** $x_4 = 1$**,** $x_5 = 0$**. Since** $x_3 \neq 0$**, we have** $y_1 = 0$**. Since** $x_4 \neq 0$**, we have** $y_2 = 0$**. Since** $x_1 \neq 0$ **we have** $y_4 = 0$**. This gives**

$$y_4 = 0 = -4 + 2y_3, \quad y_5 = -5 + y_3,$$

which gives $y_3 = 2$ **and** $y_5 = -3$**. Since** $y_5 < 0$**, this solution is not optimal.**

(b) $x_1 = 0$, $x_2 = 4$;

Answer: This gives $x_3 = 0$**,** $x_4, x_5 > 0$ **and hence** $y_2 = y_3 = y_5 = 0$**, giving**

$$y_5 = 0 = -5 + 2y_1, \quad y_4 = -4 + y_1,$$

so $y_1 = 5/2$ **and** $y_4 = -3/2$**. Since** $y_4 < 0$**, this solution is impossible.**

Continued on page 5

[8] **4.** Consider the LP: maximize x_1 subject to $x_1 \leq x_2$, $x_2 \leq x_3$, $x_3 \leq 4$, and $x_1, x_2, x_3 \geq 0$. Show how the perturbation method ensures that you increase z with every pivot. Does it matter (to the pivots that you make) whether you take $\epsilon_1 >> \epsilon_2 >> \epsilon_3$ or take the reverse?

Answer:

$$\begin{aligned} x_4 &= \epsilon_1 - x_1 + x_2 \\ x_5 &= \epsilon_2 - x_2 + x_3 \\ x_6 &= 4 + \epsilon_3 - x_3 \\ z &= x_1 \end{aligned}$$

x_1 **enters,** x_4 **leaves**

$$\begin{aligned} x_1 &= \epsilon_1 - x_4 + x_2 \\ x_5 &= \epsilon_2 - x_2 + x_3 \\ x_6 &= 4 + \epsilon_3 - x_3 \\ z &= \epsilon_1 - x_4 + x_2 \end{aligned}$$

x_2 **enters,** x_5 **leaves**

$$\begin{aligned} x_1 &= \epsilon_1 + \epsilon_2 - x_4 - x_5 + x_3 \\ x_2 &= \epsilon_2 - x_5 + x_3 \\ x_6 &= 4 + \epsilon_3 - x_3 \\ z &= \epsilon_1 + \epsilon_2 - x_4 - x_5 + x_3 \end{aligned}$$

x_3 **enters,** x_6 **leaves**

$$\begin{aligned} x_1 &= 4 + \epsilon_1 + \epsilon_2 + \epsilon_3 - x_4 - x_5 - x_6 \\ x_2 &= 4 + \epsilon_2 + \epsilon_3 - x_5 - x_6 \\ x_3 &= 4 + \epsilon_3 - x_6 \\ z &= 4 + \epsilon_1 + \epsilon_2 + \epsilon_3 - x_4 - x_5 - x_6 \end{aligned}$$

Done!

So the z value increases from 0 to ϵ_1 to $\epsilon_1 + \epsilon_2$ to $4 + \epsilon_1 + \epsilon_2 + \epsilon_3$, always increasing (at least "infinitesimally"). Since our pivoting never involved a comparison of any of the ϵ_i's to one another, the dictionaries will be the same whatever their ordering of magnitude.

Continued on page 6

Continued on page 7

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The University of British Columbia

Midterm Examinations - March 2010

Mathematics 340–201

Closed book examination | Time: 60 minutes

Name ____________________ **Signature** ____________________

Student Number ____________________ **Instructor's Name** ____________________

Section Number ____________________

Special Instructions:

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1		8
2		8
3		8
4		8
Total		32