1.(a) $x_3 = -2 + x_1 + x_2 \overset{\text{add}}{+ x_0}$ x_0 ent, $x_0 = 2 - x_1 - x_2 + x_3$ x_1 ent

$x_4 = 0 - x_1 + x_2 + x_0$ x_3 leaves $x_4 = 2 - 2x_1 + x_3$ x_4 leaves

---

$x_1 = 1 - \frac{1}{2}x_4 + \frac{1}{2}x_3$ x_2 ent $x_2 = 1 + \frac{1}{2}x_4 ~~- x_0~~ + \frac{1}{2}x_3$

$x_0 = 1 + \frac{1}{2}x_4 - x_2 + \frac{1}{2}x_3$ x_0 leaves $x_1 = 1 - \frac{1}{2}x_4 + \frac{1}{2}x_3$

$z = -x_2 = -1 - \frac{1}{2}x_4 - \frac{1}{2}x_3$ done

~~$x_4 = 2 - 2x_1 + x_3$~~ $(x_1^*, x_2^*) = (1,1)$ is optimum, with $z^* = -1$.

~~$x_2 = 2$~~

~~$z =$~~

(b) $z = -x_2$; $-1 - \frac{1}{2}x_4 - \frac{1}{2}x_3 = -1 - \frac{1}{2}(-x_1 + x_2) - \frac{1}{2}(-2 + x_1 + x_2)$

$= -1 + \frac{1}{2}x_1 - \frac{1}{2}x_2 + 1 - \frac{1}{2}x_1 - \frac{1}{2}x_2 = -x_2$.

(C) start from $x_2 = 1 + \frac{1}{2}x_4 + \frac{1}{2}x_3$ x_3 ent. ~~so unbounded.~~

$x_1 = 1 - \frac{1}{2}x_4 + \frac{1}{2}x_3$ nothing leaves Check: $z = x_1 + x_2$

$z = x_1 + x_2 = 2 + x_3$ $2 + x_3 = 2 + (-2 + x_1 + x_2) = x_1 + x_2$

2. $f = -x_1 + ax_2$, $g_1 = x_1^2 + x_2^2 - 2$, $g_2 = x_1 - 1$. Since both g_1, g_2 active at $(1,1)$:

$\nabla f + u_1 \nabla g_1 + u_2 \nabla g_2 = 0$ or $(-1, a) + u_1(2x_1, 2x_2) + u_2(1,0) = 0$

or $u_1(2,2)$

So $\left.\begin{array}{l} -1 + u_1 2 + u_2 = 0 \\ a + u_1 2 = 0 \end{array}\right\}$ so $u_1 = \frac{-a}{2}$, $u_2 = 1 - 2u_1 = 1 + 2a$.

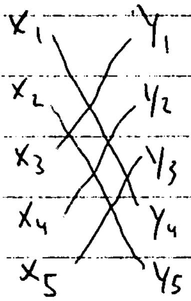
So $\frac{-a}{2}$ and $1 + 2a \geq 0$ so $-\frac{1}{2} \leq a \leq 0$.

3. min $5y_1+8y_2+8y_3$

s.t. $y_1+y_2+2y_3 \geq 4$

$y_1+2y_2+y_3 \geq 5$

$y_1, y_2, y_3 \geq 0$



$y_1 = 3 + 2y_4 - y_5 - 3y_3$

$y_2 = 1 - y_4 + y_5 + y_3$

$\omega = -23 - 2y_4 - 3y_5 - y_3$

$f(x) = -4x_1 - 5x_2$ $\quad g_1 = \overset{x_1+x_2-5}{\cancel{5\quad x_2}}$, $g_2 = x_1+2x_2-8$, $g_3 = 2x_1+x_2-8$, $g_4 = -x_1$, $g_5 = -x_2$.

(2,3): g_1, g_2 active, so $\nabla f + u_1 \nabla g_1 + u_2 \nabla g_2 = 0$

or $(-4,-5) + u_1(1,1) + u_2(1,2) = 0$ solution: $u_1 = 3, u_2 = 1$.

~~(u_1, u_2) are just the optimal solution in the corresponding (y_1, y_2) dual LP, and (u_3, u_4, u_5) are just (y_3, y_4, y_5) in that solution (which are zero).~~

$(u_1, u_2, \ldots, u_5)$ correspond to $(x_3, x_4, x_5, x_1, x_2)$ that corresp to $(y_1, \ldots, y_5)$, and the KKT multipliers $(u_1, \ldots, u_5) = (3,1,0,0,0)$ at (2,3) is just the optimal solution of the dual LP $(y_1, \ldots, y_5) = (3,1,0,0,0)$.

4. (a) $B = \begin{array}{c|cc} & \text{candy} & \text{capp.} \\ \text{g sugar} & 40 & 20 \\ \$ & 1 & 2 \end{array}$, $B^{-1} = \begin{array}{c|cc} & \text{g sug} & \$ \\ \text{candy} & 1/30 & -1/3 \\ \text{capp} & -1/160 & 2/3 \end{array}$

(b) candy bars per grams of sugar. We would consume $\frac{1}{3}$ ~~four~~ of a candy bar~~s~~ less, and $\frac{2}{3}$ of a cappuccino more.

(c) New z row is $z = 600 + 10x_2 - \frac{5}{2}x_4 - 50x_5$, so x_2 enters, x_3 leaves

$x_2 = \frac{9}{2} - \frac{3}{2}x_3 + \frac{1}{40}x_4 - x_5$

$x_1 = \frac{5}{2}$ + irrelevant

$z = 645 - 15x_3 - \frac{9}{4}x_4 - 60x_5$

$(x_1, x_2, x_3) = (\frac{5}{2}, \frac{9}{2}, 0)$

4(d) $b = \begin{bmatrix} 100 \\ b_2 \end{bmatrix}$, $B^{-1}b = \begin{bmatrix} \frac{100}{30} - b_2/3 \\ -\frac{100}{60} + 2b_2/3 \end{bmatrix} = \begin{bmatrix} \frac{10}{3} - b_2/3 \\ -\frac{5}{3} + 2b_2/3 \end{bmatrix}$.

$$c_B B^{-1} b = \begin{bmatrix} 150 & 150 \end{bmatrix} \begin{bmatrix} \ \end{bmatrix} = 250 + 50 b_2 .$$

Dictionary:

$$x_1 = \frac{10}{3} - \frac{b_2}{3} + \frac{1}{3} x_2 - \dots$$

$$x_3 = -\frac{5}{3} + \frac{2b_2}{3} - \frac{2}{3} x_2 + \dots$$

$$z = 250 + 50 b_2 \quad - \dots$$

(e) $\boxed{\frac{5}{2} \le b_2 \le 10 : \quad z = 250 + 50 b_2 \quad (x_1, x_2, x_3) = \left(\frac{10}{3} - \frac{b_2}{3}, 0, -\frac{5}{3} + \frac{2b_2}{3}\right)}$

b_2 slightly > 10: x_1 leaves, x_2 enters, get

$$x_2 = b_2 - 10 + \text{etc.}$$

$$x_3 = -\frac{5}{3} + \frac{2b_2}{3} - \frac{2}{3}(b_2 - 10) + \text{etc.}$$

$$[illegible] = 5 + \text{etc.}$$

$$z = 250 + 50 b_2$$

$$\boxed{\begin{gathered} \text{So: } \quad b_2 \ge 10 : \ z = 250 + 50 b_2 \\ (x_1, x_2, x_3) = (0, b_2 - 10, 5) \end{gathered}}$$

b_2 slightly $< \frac{5}{2}$: x_3 leaves, x_4 enters

$$x_4 = 20(5 - 2b_2) + \text{etc.}$$

$$x_1 = \frac{10}{3} - \frac{b_2}{3} - \frac{20}{30}(5 - 2b_2) + \text{etc.}$$

$$= b_2 + \text{etc.}$$

$$z = 250 + 50 b_2 - \frac{5}{2} 20(5 - 2b_2) + \text{etc.}$$

$$= 150 b_2 + \text{etc.}$$

$$\boxed{\begin{gathered} 0 \le b_2 \le \frac{5}{2} : \ z = 150 b_2 \\ (x_1, x_2, x_3) = (b_2, 0, 0) \end{gathered}}$$

For $b_2 > 1000$ you have a lot of money, and soda has no sugar so you can buy as much as you want without violating the sugar constraint. Your main constraint is sugar, so you buy capps until your sugar limit is reached. For $b_2 < .001$ you buy candy bars, the best caffeine / \$ value since the sugar limit is no consideration

(f) Change is in $A_N = \begin{bmatrix} t & \cdot & \cdot \\ 1 & \cdot & \cdot \end{bmatrix}$, and only in x_2 coefs. These coefs are:

$$\begin{pmatrix} x_1 \\ x_3 \end{pmatrix} \text{ rows: } -B^{-1}\begin{bmatrix} t \\ 1 \end{bmatrix} x_2 = -\begin{bmatrix} t/30 - 1/3 \\ -t/60 + 2/3 \end{bmatrix} x_2$$

and

$$z \text{ row: } \left(c_2 - [c_1\ c_3]\begin{bmatrix} \\ \end{bmatrix}\right) x_2$$

$$= \left(50 - [150\ 150]\begin{bmatrix} \\ \end{bmatrix}\right) x_2 = \left(50 \overset{-\frac{5t}{2}}{-150}\right) x_2 = \cancel{0} - \left(\frac{5t}{2}\right) x_2$$

Dictionary final for all $t \geq 0$. Since soda with $t=0$ isn't used in final solution, it certainly won't be used if it has more sugar.

5. See solutions to homework #6