Math 340, Dec '99

1\. (a) $x^* = (\frac{5}{2}, \frac{3}{2}, 0)$ has

$\frac{5}{2} + \frac{3}{2} + 2\cdot 0 \le 4$ yes

$\frac{5}{2}\cdot 2 \quad + 3\cdot 0 \le 5$ yes

$\frac{5}{2}\cdot 2 + \frac{3}{2} + 3\cdot 0 \le 7$ yes

and $x^* \ge 0$.

(b) $2(x_1 + x_2 + 2x_3 \le 4)$

$\frac{1}{2}(2x_1 \quad + 3x_3 \le 5)$

$3x_1 + 2x_2 + \frac{11}{2}x_3 \le \frac{21}{2}$ so clearly $3x_1 + 2x_2 + 4x_3 \le 21/2$ also.

2\. (a)

$x_3 = -1 + x_1 + 2x_2 + x_0$

$x_4 = 1 - x_1 + x_2 + x_0$

$\omega = \qquad - x_0$

x_0 enters, x_3 leaves

$x_0 = 1 - x_1 - 2x_2 + x_3$

$x_4 = 2 - 2x_1 - x_2 + x_3$

x_2 enters, x_0 leaves

$x_2 = \frac{1}{2} - \frac{1}{2}x_1 - \cancel{\frac{1}{2}x_0} + \frac{1}{2}x_3$

$x_4 = \frac{3}{2} - \frac{3}{2}x_1 + \cancel{\frac{1}{2}x_0} + \frac{1}{2}x_3$

$z = \quad \cancel{x_1} + 3x_2$

$= \frac{3}{2} - \frac{1}{2}x_1 \qquad + \frac{3}{2}x_3$

x_3 enters, nothing leaves

Problem is unbounded

(b) $x_1 = 0$

$x_3 = 1000$ in final dict gives

$z = 1500 + 3/2$

$x_2 = \frac{1}{2} + \frac{1}{2}\cdot 1000 = 500 + \frac{1}{2}$

Ans: $x^* = (0, 500 + \frac{1}{2})$

3. First Step: $B_1 = I$. $x_B = \{x_4, x_5, x_6\}$, $x_N = \{x_1, x_2, x_3\}$

$y^T = c_B^T B^{-1} = [0\ 0\ 0]$, x_N coef in z row:

$$\left(c_N^T - c_B^T B^{-1} A_N\right) = [3\ \ 2\ \ 4] \qquad t = 5/3 \quad \text{new } x_B^* = \begin{bmatrix} 2/3 \\ 5/3 \\ 2 \end{bmatrix}$$

so x_3 enters. $Bd = a_3 = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$, $d = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$, $x_B^* = \begin{bmatrix} 4 \\ 5 \\ 7 \end{bmatrix}$ so x_5 leaves

$$B_2 = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 3 & 1 \end{bmatrix} \qquad x_B = \{x_4, x_3, x_6\},\ x_N = \{x_1, x_2, x_5\}$$

$y^T B = c_B^T = [0\ 4\ 0]$, $y^T = [0\ \ 4/3\ \ 0]$,

$$c_N^T - y^T A_N = [3\ 2\ 0] - [0\ \ 4/3\ \ 0]\begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} = [3\ 2\ 0] - [8/3\ \ 0\ \ 4/3]$$

$= [1/3\ \ 2\ \ -4/3]$ so x_2 enters. $Bd = a_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $d = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$x_B^* = \begin{bmatrix} 2/3 \\ 5/3 \\ 2 \end{bmatrix}$, so $t = \frac{2}{3}$, x_4 leaves. New $x_B^* = \begin{bmatrix} 2/3 \\ 5/3 \\ 4/3 \end{bmatrix}$

4. $B^{-1} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \\ -3 & 1 & 1 \end{bmatrix}$, so $B^{-1}b = B^{-1}\begin{bmatrix} u \\ 8 \\ 8 \end{bmatrix} = \begin{bmatrix} 2u-8 \\ 8-u \\ 16-3u \end{bmatrix}$

$$c_B^T B^{-1} b = [4\ 5\ 0]\begin{bmatrix} - \\ - \\ - \end{bmatrix} = 4(2u-8)+5(8-u) = 8+3u.$$

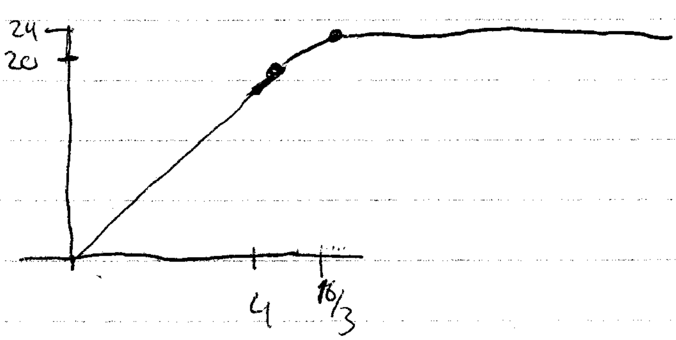
Optimal if feasible, i.e. $2u-8, 8-u, 16-3u \geq 0$ or $4 \leq u \leq 16/3$.

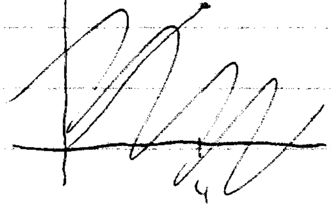
For slightly $u > 16/3$, x_5 leaves, x_3 enters in a dual simplex pivot:

$$\begin{aligned} x_1 &= 2u-8-2x_3+x_4 \\ x_2 &= 8-u+x_3-x_4 \\ x_5 &= 16-3u+3x_3-x_4 \\ z &= 8+3u-3x_3-x_4 \end{aligned} \quad \curvearrowright \quad \begin{aligned} x_1 &= 8/3 + \cdots \\ x_2 &= 8/3 + \cdots \\ x_3 &= \frac{3u-16}{3} + \frac{1}{3}x_5 + \frac{1}{3}x_4 \\ z &= 24 + \cdots \end{aligned}$$

For u slightly < 4 x_1 leaves, x_4 enters

$$\begin{aligned} x_4 &= 8-2u+2x_3+x_1 \\ x_2 &= u + \cdots \\ x_5 &= 8-u \\ z &= 5u \end{aligned}$$





5. (a) \$3 per microprocessor, \$1 / minute labour, \$0 / heating element.

(b) yes ; no.

(c) Marginal cost = $3 \cdot 1 + 1 \cdot \frac{1}{2} = \$3\frac{1}{2} > $ profit, so no.

(d) $\cancel{B^{-1}a_6 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \\ -3 & \cdot & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \cdot \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}}$

$\cancel{c_6 - c_B^T B^{-1} a_6 = 6 - [4\ 5\ 0] \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = 2}$

CHANGED

$\cancel{X_6}$

$$X_1 + X_2 \leq 4 \quad \Rightarrow \quad X_6 = 4 - X_1 - X_2$$

$$X_6 = 4 - X_1 - X_2 = -1 + X_3$$

rest of dict same

X_6 leaves, X_3 enters   dual pivot

$X_3 = 1 + X_6$

$X_1 = 0 - \ldots$

$X_2 = 4 + \ldots$

$X_5 = 4 + \ldots$

$z = 20 - \ldots$

So done.

6. $\nabla f = \left(2(x_1-5),\ 2(x_2-1)\right)$ $\qquad g = \begin{bmatrix} x_1+x_2-2 \\ -x_1 \\ -x_2 \end{bmatrix}$

At $(2,0)$ g_1 and g_3 active, so

$$\nabla f + u_1 \nabla g_1 + u_3 \nabla g_3 = 0$$

or

$$\left(2(x_1-5),\ 2(x_2-1)\right) + u_1(1,1) + u_3(0,-1) = 0$$

or

$$(-4,2) + u_1(1,1) + u_3(0,-1) = 0$$

or

$$\left.\begin{aligned} u_1 &= 4 \\ u_1 - u_3 &= -2 \end{aligned}\right\} \quad u_1 = 4,\ u_3 = 6$$

So $(2,0)$ is a KKT point and we are done, BECAUSE any KKT point in a convex program is a global minimum (and f and g are convex!).