Marks

[8] **1.** Find the value of "Alice announces a pure strategy" and "Betty announces a pure strategy" for the matrix game

$$A = \begin{bmatrix} 1 & 4 \\ 5 & 2 \end{bmatrix} .$$

Find the value of the mixed strategy games and the optimal mixed strategies for Alice and Betty.

Duality gap! Betty Pure 4 > Alice Pure 2

solve

$$[x_1 \quad 1-x_1]\begin{bmatrix} 1 & 4 \\ 5 & 2 \end{bmatrix} = [v \quad v]$$

$$1\cdot x_1 + 5(1-x_1) = v = 4x_1 + 2(1-x_1)$$

$$5 - 4x_1 = 2 + 2x_1$$

$$3 = 6x_1, \quad x_1 = \frac{1}{2}$$

Alice optimal mixed: $[\frac{1}{2} \quad \frac{1}{2}]$, $v = 2 + 2x_1 = 2 + 2\cdot\frac{1}{2} = 3$

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[10] **2.** Consider the problem: maximize $x_1 + 2x_2$ subject to $x_1 - 2x_2 \geq 4$, $x_1 - 2x_2 \geq 7$, $x_1 + x_2 \leq 1$, and $x_1, x_2 \geq 0$. Write this as a linear program in standard form. Use the two-phase method, **adding an auxilliary variable x_0 to EVERY slack variable equation in the dictionary,** to solve this LP. Use the smallest subscript rule to break any ties for entering or leaving variables. [Hint: This should not take more than two or three pivots.]

Done in many past midterms
and homework (with different numbers)

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[10] **3.** Consider the LP: maximize x_1+18x_2 subject to $x_1+x_2 \le 5$, $x_1+9x_2 \le 9$, $3x_1+x_2 \le 9$, and $x_1, x_2 \ge 0$.

(a) Write down the dual linear program.

(b) Use complementary slackness to check if $x_1 = 3$, $x_2 = 0$ is an optimal solution for this linear program.

(c) Use complementary slackness to check if $x_1 = 0$, $x_2 = 1$ is an optimal solution for this linear program.

max x_1+18x_2

slack

s.t.

$x_3 \to (x_1+x_2 \le 5)$ y_1

$x_4 \to (x_1+9x_2 \le 9)$ y_2

$x_5 \to (3x_1+x_2 \le 9)$ y_3

$$\underbrace{(y_1+\dots)x_1 + (y_1+\dots)}_{} x_2 \le (5y_1+\dots)$$

must be an upper bound for x_1+18x_2 so

minimize $5y_1+9y_2+9y_3$ (or maximize $-5y_1-9y_2-9y_3$)

slack

s.t.

$y_4 \to 1 \le y_1+y_2+3y_3$

$y_5 \to 18 \le 5y_1+9y_2+9y_3$ (and $y_1, y_2, y_3 \ge 0$)

$x_3 \leftrightarrow y_1$

$x_4 \leftrightarrow y_2$

$x_5 \leftrightarrow y_3$

$x_1 \leftrightarrow y_4$

$x_2 \leftrightarrow y_5$

(b) $x_1=3, x_2=0$ gives

$x_3 = 2 \Rightarrow y_1=0$

$x_4 = 6 \Rightarrow y_2 = 0$

$x_5 = 0$

$x_1 = 3 \Rightarrow y_4 = 0$

$x_2 = 0$

so $y_4 = \overset{0}{\cancel{y_1}} + \overset{0}{\cancel{y_2}} + 3y_3 - 1$

$0 = 3y_3 - 1$

so $y_3 = 1/3$.

Then $y_5 = 5\cdot 0 + 9\cdot 0 + 9\cdot\frac{1}{3} - 18 < 0$

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so (b) proposed solution is not optimal

[6] **4.** Consider the linear program: maximize $s + 10t + r$ subject to $2s + 5t + r \leq 16$, $3s + t - 3r \leq 12$, $4s + 3t + 3r \leq 20$, and $t, r \geq 0$, but where s can be negative, zero, or positive. Replace s by two variables, namely $s = s_1 - s_2$, where we impose the condition $s_1, s_2 \geq 0$.

(a) The three constraints of this LP can be written as

this is covered in 2015

$$[\text{“big } A\text{”}] \begin{bmatrix} s_1 \\ s_2 \\ t \\ r \\ w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 16 \\ 12 \\ 20 \end{bmatrix}$$

where w_1, w_2, w_3 are slack variables. Write out the matrix “big A.”

(b) Can both s_1 and s_2 be basic in some dictionary of the simplex method? Justify your answer using (part of) “big A.”

not really covered in 2015

first slack $= 16 - 2s - 5t - r$

$= 16 - 2s_1 + 2s_2 - 5t - r$

big A

$$\begin{array}{cccc|ccc} s_1 & s_2 & t & r & w_1 & w_2 & w_3 \\ 2 & -2 & 5 & 1 & 1 & 0 & 0 \\ & & & & 0 & 1 & 0 \\ & & & & 0 & 0 & 1 \end{array}$$

similarly last 2 rows

$$\left[\begin{array}{cccc|c} 2 & -2 & 5 & 1 & \\ 3 & -3 & 1 & -3 & \text{Identity} \\ 4 & -4 & 3 & 3 & \end{array}\right]$$

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[32] **5.** (4 points for each part) Briefly justify your answers:

(a) In the matrix game

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 5 & 4 & 1 \end{bmatrix}$$

someone claims that Alice's optimal mixed strategy is [1/2 1/2 0] and that Betty's is [1/2 1/2 0]. Make a quick calculation (without using the simplex method) to determine whether or not this is true; explain why your quick calculation works.

$[1/2\ 1/2\ 0]\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 5 & 4 & 1 \end{bmatrix} = [3/2\ \ 3/2\ \ 1]$ ← Betty chooses 1. Hence value ≥ 1

$\begin{bmatrix} & & \end{bmatrix}\begin{bmatrix} 1/2 \\ 1/2 \\ 0 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 3/2 \\ 9/2 \end{bmatrix}$ ← Alice chooses 9/2 Hence value $\leq \frac{9}{2}$

But don't match

(b) Consider a $n \times n$ weighted bipartite matching problem (Assignment Problem), where $n \geq 6$ and where x_{ij} is 1 if person i does task j (entirely). Argue that $x_{16}, x_{36}, x_{34}, x_{14}$ cannot all be basic in any dictionary of the simplex method.

Not done in 2015

(c) Give an example of a linear program whose first three pivots must be degenerate. How do you know that these first three pivots must be degenerate?

max x_1

s.t. $0 \leq x_1 \leq x_2 \leq x_3 \leq x_4 \leq 9$

in first BFS, $x_1 = x_2 = x_3 = x_4 = 0$

~~So before x_1 or x_2 [illegible] x_3 must [illegible]~~

So x_1 enters but $x_1 \leq x_2$ ← 0 in 1st dictionary still 0

now $x_4 = x_2 - x_1$ ⌒ $x_1 = x_2 - x_4$

now $z = x_2 - x_4$

So x_2 enters, 2nd slack leaves

Third pivot x_3 enters, 3rd slack leaves

etc.

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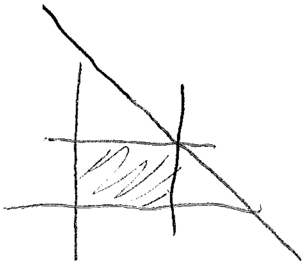
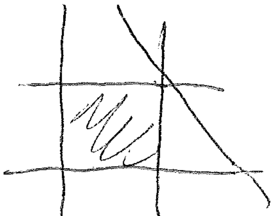
(d) Give an example of a linear program such that the perturbation applied to it requires no more than two pivots, but if we change the order of the rows (or equivalently change the order of $\epsilon_1, \epsilon_2, \ldots$) the simplex method can take three iterations. Justify your claim, either in words or pictures.

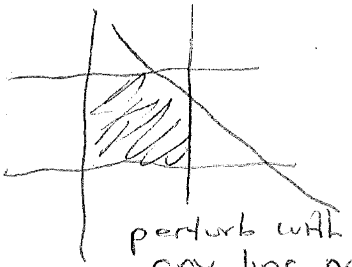
See homework. The two perturbed problems look like:



original



perturb with slanted line moving the most



perturb with any line not slanted moving most

(e) Consider a linear program that has various constraints including $x_1 - 2x_2 \geq 4$ and $x_1 - 2x_2 \geq 7$. Can both of the slack variables corresponding to these two constraints be nonbasic in some dictionary of the simplex method? Explain.

$$x_1 - 2x_2 \geq 4 \quad \rightsquigarrow \quad -x_1 + 2x_2 \leq -4 \quad \leftarrow x_{12}$$

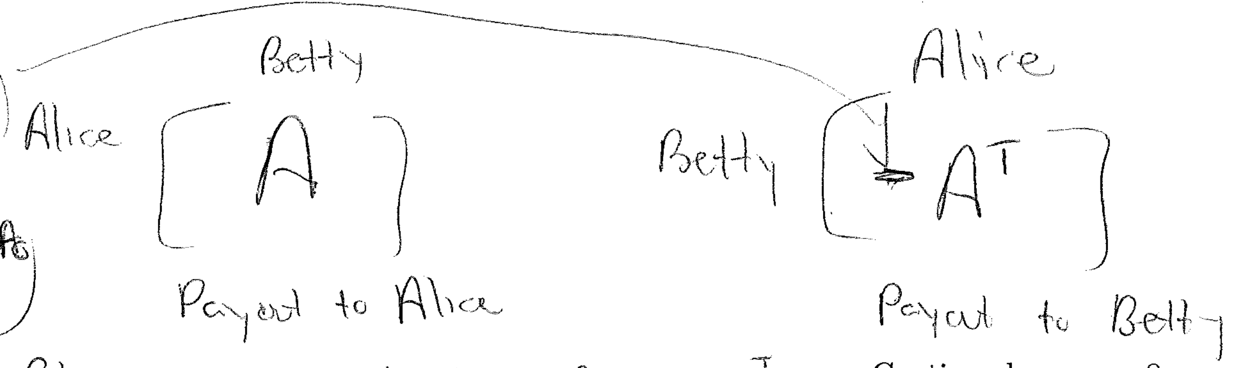
$$x_1 - 2x_2 \geq 7 \quad \rightsquigarrow \quad -x_1 + 2x_2 \leq -7 \quad \leftarrow x_{13}$$

in any feasible solution $x_1 - 2x_2 \geq 7$

so $x_1 - 2x_2 - 4 \geq 0 \leftarrow x_{12}$

(f) In this class we viewed an $m \times n$ matrix, A, as a matrix game giving the payout to Alice with Alice playing m pure strategies represented by A's rows, and Betty playing n pure strategies columns represented by A's columns. What matrix would you get if Alice and Betty exchange roles? Explain.

(1) Payout to Betty is − Payout to Alice



(2) Row & Cols are exchanged to $A \rightarrow A^T$

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(g) Find the value of the mixed strategy games for the matrix game

$$A = \begin{bmatrix} -1 & -4 & -9 & -16 & -25 & -36 & -49 \\ -49 & -36 & -25 & -16 & -9 & -4 & -1 \end{bmatrix}.$$

Justify your answer.

Opposite game (Alice & Betty reversed) is

$$-A^T = \begin{bmatrix} 1 & 49 \\ 4 & 36 \\ 9 & \vdots \\ \vdots & \vdots \\ 49 & 1 \end{bmatrix}.$$

Since $-A^T$ has convex columns, this reduces to $\begin{bmatrix} 1 & 49 \\ 49 & 1 \end{bmatrix}$. So original game is $\begin{bmatrix} -1 & -49 \\ -49 & -1 \end{bmatrix}$. So best mixed strategy is 50%-50% for both, value is -25.

(h) Consider the line $y = a + bx$ which is the best "max approximation" regression line to the data points $(0, 4)$, $(1, 6)$, $(2, 7)$, $(3, 10)$, $(5, 11)$, i.e., such that

$$d = \max(|4 - a|, |6 - a - b|, |7 - a - 2b|, |10 - a - 3b|, |11 - a - 5b|)$$

is minimized. Assume that you know that at optimality (i.e., when d is minimized), a, b, d are all positive. At the optimal a, b, d, at least how many of

$$|4 - a|, |6 - a - b|, |7 - a - 2b|, |10 - a - 3b|, |11 - a - 5b| \quad \leftarrow (*)$$

must equal d? What could you say if you had ten data points instead of five?

This can be written as

maximize d s.t.

$-d \leq 4 - a \leq d$

$-d \leq 6 - a - b \leq d$

$\vdots$

So have 3 decision variables 10 slack variables,

So at optimal solution have 3 non-basic variables,

So 3 variables equal to zero

Since $a, b, d > 0$ in optimal solution, 3 slack variables are zero. Since $d > 0$, we cannot have

$-d = \text{anything} = d$,

hence the three zero variables are slack variables corresponding to different points. Hence at least three of $(*)$ must equal zero.

Same conclusion with 10 data points, since we still have 3 non-basic variables

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