Marks

[10] **1.** Consider the problem

$$\text{maximize} \quad 3x_1 + 2x_2 + 4x_3 \quad \text{subject to} \quad x_1, x_2, x_3 \geq 0,$$

$$\begin{array}{llll} x_1 & +x_2 & +2x_3 & \leq 4, \\ 2x_1 & & +3x_3 & \leq 5, \\ 2x_1 & +x_2 & +3x_3 & \leq 7. \end{array}$$

A friend thinks she solved it using the simplex method, with a final dictionary of

$$\begin{array}{lllllll} x_1 & = & 5/2 & +?x_3 & +?x_4 & +?x_5 \\ x_2 & = & 3/2 & +?x_3 & +?x_4 & +?x_5 \\ x_6 & = & 1/2 & +?x_3 & +?x_4 & +?x_5 \\ z & = & 21/2 & -(3/2)x_3 & -2x_4 & -(1/2)x_5, \end{array}$$

where coefficients indicated with ?'s have coffee stains all over them (i.e. are unknown to you). However, your friend is not sure that this partial final dictionary is correct. Using only this limited information on the final dictionary,

(a) give a feasible $x^* = (x_1^*, x_2^*, x_3^*)$ for which the objective is 21/2 (verify this x^* is feasible), and

(b) prove that for any feasible x^* the objective must be at most 21/2.

[15] **2.** Consider the problem

$$\text{maximize} \quad x_1 + 3x_2 \quad \text{subject to} \quad x_1, x_2 \geq 0,$$

$$\begin{array}{lll} -x_1 & -2x_2 & \leq -1, \\ x_1 & -x_2 & \leq 1. \end{array}$$

(a) Use the two-phase method to solve this problem, with the largest coefficient rule to choose the entering variable, breaking ties by choosing the smallest subscript. Clearly state which variable enters and which leaves on each pivot, and what your final dictionary means in terms of the original problem.

(b) Use your final dictionary to find a feasible $x^* = (x_1^*, x_2^*)$ with $z^* = x_1^* + 3x_2^* = 1500 + (3/2)$.

[20] **3.** Use the revised simplex method with the highest coefficient rule to begin solving the linear program in problem 1. Stop after the second iteration (i.e. first you will find x_3 enters and x_5 leaves; second you will find x_2 enters and x_4 leaves; then you stop). Be sure you find x_B^* for the third iteration (which is done at the end of the second iteration).

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[20] **4.** The linear program given below on the left, which involves a parameter, u, has its final dictionary for $u = 5$ (our favourite problem) given below on the left.

$$\begin{array}{llll} \text{max.} & 4x_1 + 5x_2 & \text{s. t.} & x_1, x_2 \geq 0, \\ & x_1 \quad +x_2 & \leq u, & \\ & x_1 \quad +2x_2 & \leq 8, & \\ & 2x_1 \quad +x_2 & \leq 8. & \end{array} \qquad \begin{array}{rcrrr} x_1 & = & 2 & -2x_3 & +x_4 \\ x_2 & = & 3 & +x_3 & -x_4 \\ x_5 & = & 1 & +3x_3 & -x_4 \\ z & = & 23 & -3x_3 & -x_4 \end{array}$$

(a) Determine B^{-1} for the above dictionary directly from the dictionary.

(b) Use B^{-1} and the formulas at the end of this exam to find the above dictionary as a function of u (i.e. not specialized to $u = 5$).

(c) Determine optimal solutions of the linear program for all values of $u \geq 0$. If $z^*(u)$ denotes the optimum of the linear program (as a function of u), plot $z^*(u)$.

[20] **5.** To build a "dual coffee maker" we require one microprocessor, one minute of labour, and two heating elements; such a machine yields \$4 of profit per machine. An "espresso maker" requires one microprocessor, two minutes of labour, and one heating element; this machines yields \$5 of profit per machine. As resources we have 5000 microprocessors, 8000 minutes of labour, and 8000 heating elements. This suggests the linear program of problem 4, with $u = 5$, whose final dictionary is given there (with x_1, x_2 representing the number of machines in thousands).

(a) Given the final dictionary and corresponding optimal solution, what are the marginal values of one microprocessor, one minute of labour, and one heating element?

(b) Someone offers to sell us a few microprocessors for \$2.75 each. Will we buy some? How about for \$4 each?

(c) A soup maker can be made using one microprocessor, 1/2 of a minute of labour, and one heating element. We can sell such a machine for \$2 profit. Is it profitable to make such machines?

(d) Some law prohibits you from producing more than 4000 machines total in your factory. Add this new constraint to the current dictionary, and use dual simplex pivot(s) to find the new optimal solution. (Note: the soup maker is not considered in this part of the problem.)

[15] **6.** Let T be the region where $x_1, x_2 \geq 0$ and $x_1 + x_2 \leq 2$. Show that $(2, 0)$ is the point in T minimizing $f(x) = (x_1 - 5)^2 + (x_2 - 1)^2$, using KKT theory. What basic fact about convex programming helps you solve this problem?

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Formulae

The following formulae may be of use. They will not be explained; you are assumed to understand what they mean and to what they refer.

$$\begin{array}{rll} x_B = & B^{-1}b & -B^{-1}A_N x_N \\ z = & c_B^T B^{-1} b & +(c_N^T - c_B^T B^{-1} A_N) x_N \end{array}$$

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$$y^T = c_B^T B^{-1}, \qquad y^T B = c_B^T, \qquad d = B^{-1} a, \qquad Bd = a, \qquad x_B^* - td$$

$$B_{i+1} = B_i E_i, \qquad \text{with } E_i \text{ formed using } d \text{ from } i\text{-th dictionary}$$

$$\min f(x), \text{ s.t. } g(x) \le 0, \qquad \nabla f + u_1 \nabla g_1 + \cdots + u_n \nabla g_n = 0$$

$$u_i \ge 0, \qquad u_i = 0 \text{ if } g_i \text{ is inactive.}$$

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Be sure that this examination has 4 pages including this cover

The University of British Columbia

Final Examinations - December 1999

Mathematics 340–102

Closed book examination

Time: $2\frac{1}{2}$ hours

Name ______________________ **Signature** ______________________

Student Number ______________________ **Instructor's Name** ______________________

Section Number ______________________

Special Instructions:

Candidates may not use any notes or calculators. A list of formulae is provided at the end of this exam. Answer questions in the booklets provided.

Rules governing examinations

1. Each candidate should be prepared to produce his library/AMS card upon request.

2. Read and observe the following rules:

No candidate shall be permitted to enter the examination room after the expiration of one half hour, or to leave during the first half hour of the examination.

Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.

CAUTION - Candidates guilty of any of the following or similar practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.

(a) Making use of any books, papers or memoranda, other than those authorized by the examiners.

(b) Speaking or communicating with other candidates.

(c) Purposely exposing written papers to the view of other candidates. The plea of accident or forgetfulness shall not be received.

3. Smoking is not permitted during examinations.

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Total		100