Marks

[8] **1.** Find the value of "Alice announces a pure strategy" and "Betty announces a pure strategy" for the matrix game

$$A = \begin{bmatrix} 1 & 4 \\ 5 & 2 \end{bmatrix} .$$

Find the value of the mixed strategy games and the optimal mixed strategies for Alice and Betty.

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[10] **2.** Consider the problem: maximize $x_1 + 2x_2$ subject to $x_1 - 2x_2 \geq 4$, $x_1 - 2x_2 \geq 7$, $x_1 + x_2 \leq 1$, and $x_1, x_2 \geq 0$. Write this as a linear program in standard form. Use the two-phase method, **adding an auxilliary variable x_0 to EVERY slack variable equation in the dictionary,** to solve this LP. Use the smallest subscript rule to break any ties for entering or leaving variables. [Hint: This should not take more than two or three pivots.]

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[10] **3.** Consider the LP: maximize x_1+18x_2 subject to $x_1+x_2 \leq 5$, $x_1+9x_2 \leq 9$, $3x_1+x_2 \leq 9$, and $x_1, x_2 \geq 0$.

(a) Write down the dual linear program.

(b) Use complementary slackness to check if $x_1 = 3$, $x_2 = 0$ is an optimal solution for this linear program.

(c) Use complementary slackness to check if $x_1 = 0$, $x_2 = 1$ is an optimal solution for this linear program.

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[6] **4.** Consider the linear program: maximize $s + 10t + r$ subject to $2s + 5t + r \leq 16$, $3s + t - 3r \leq 12$, $4s + 3t + 3r \leq 20$, and $t, r \geq 0$, but where s can be negative, zero, or positive. Replace s by two variables, namely $s = s_1 - s_2$, where we impose the condition $s_1, s_2 \geq 0$.

(a) The three constraints of this LP can be written as

$$[\text{“big } A\text{”}] \begin{bmatrix} s_1 \\ s_2 \\ t \\ r \\ w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 16 \\ 12 \\ 20 \end{bmatrix}$$

where w_1, w_2, w_3 are slack variables. Write out the matrix “big A.”

(b) Can both s_1 and s_2 be basic in some dictionary of the simplex method? Justify your answer using (part of) “big A.”

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[32] **5.** (4 points for each part) Briefly justify your answers:

(a) In the matrix game

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 5 & 4 & 1 \end{bmatrix}$$

someone claims that Alice's optimal mixed strategy is [1/2 1/2 0] and that Betty's is [1/2 1/2 0]. Make a quick calculation (without using the simplex method) to determine whether or not this is true; explain why your quick calculation works.

(b) Consider a $n \times n$ weighted bipartite matching problem (Assignment Problem), where $n \geq 6$ and where x_{ij} is 1 if person i does task j (entirely). Argue that $x_{16}, x_{36}, x_{34}, x_{14}$ cannot all be basic in any dictionary of the simplex method.

(c) Give an example of a linear program whose first three pivots must be degenerate. How do you know that these first three pivots must be degenerate?

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(d) Give an example of a linear program such that the perturbation applied to it requires no more than two pivots, but if we change the order of the rows (or equivalently change the order of $\epsilon_1, \epsilon_2, \ldots$) the simplex method can take three iterations. Justify your claim, either in words or pictures.

(e) Consider a linear program that has various constraints including $x_1 - 2x_2 \geq 4$ and $x_1 - 2x_2 \geq 7$. Can both of the slack variables corresponding to these two constraints be nonbasic in some dictionary of the simplex method? Explain.

(f) In this class we viewed an $m \times n$ matrix, A, as a matrix game giving the payout to Alice with Alice playing m pure strategies represented by A's rows, and Betty playing n pure strategies columns represented by A's columns. What matrix would you get if Alice and Betty exchange roles? Explain.

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(g) Find the value of the mixed strategy games for the matrix game

$$A = \begin{bmatrix} -1 & -4 & -9 & -16 & -25 & -36 & -49 \\ -49 & -36 & -25 & -16 & -9 & -4 & -1 \end{bmatrix}.$$

Justify your answer.

(h) Consider the line $y = a + bx$ which is the best "max approximation" regression line to the data points $(0, 4)$, $(1, 6)$, $(2, 7)$, $(3, 10)$, $(5, 11)$, i.e., such that

$$d = \max(|4 - a|, |6 - a - b|, |7 - a - 2b|, |10 - a - 3b|, |11 - a - 5b|)$$

is minimized. Assume that you know that at optimality (i.e., when d is minimized), a, b, d are all positive. At the optimal a, b, d, at least how many of

$$|4 - a|, |6 - a - b|, |7 - a - 2b|, |10 - a - 3b|, |11 - a - 5b|$$

must equal d? What could you say if you had ten data points instead of five?

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Be sure that this examination has 10 pages including this cover

The University of British Columbia

Midterm Examinations - December 2014

Mathematics 340

Closed book examination

Time: $2\frac{1}{2}$ hours

Name ______________________

Signature ______________________

Student Number ______________________

Instructor's Name ______________________

Section Number ______________________

Special Instructions:

Calculators, notes, or other aids may not be used. Answer questions on the exam. This exam is two-sided!

Rules governing examinations

1. Each candidate should be prepared to produce his library/AMS card upon request.

2. Read and observe the following rules:

No candidate shall be permitted to enter the examination room after the expiration of one half hour, or to leave during the first half hour of the examination.

Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.

CAUTION - Candidates guilty of any of the following or similar practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.

(a) Making use of any books, papers or memoranda, other than those authorized by the examiners.

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(c) Purposely exposing written papers to the view of other candidates. The plea of accident or forgetfulness shall not be received.

3. Smoking is not permitted during examinations.

1		8
2		10
3		10
4		6
5		32
Total		66

Math 340 Note Sheet for the Final Exam, Fall 2014

$$\text{ValueAliceAnnouncesPure}(A) = \max_i \text{MinEntry of } i\text{-th row of } A = \max_i \min_j a_{ij}$$

$$\text{ValueBettyAliceAnnouncesPure}(A) = \min_j \text{MaxEntry of } j\text{-th column of } A = \min_j \max_i a_{ij}$$

$$\text{DualityGap} = (\text{ValueBettyAnnouncesPure}) - (\text{ValueAliceAnnouncesPure})$$

The value of Alice announces a mixed strategy is

$$\max_{\vec{p} \text{ stoch}} \text{MinEntry}(\vec{p}^{\mathrm{T}} A)$$

Is given by LP

$$\max v \quad \text{s.t.} \quad \vec{p}^T A \geq [v \ v \ \ldots \ v],$$

$$p_1 + \cdots + p_m = 1, \quad p_1, p_2, \ldots, p_m \geq 0.$$

If all entries of A are positive this is equivalent to

$$\max v \quad \text{s.t.} \quad \vec{p}^T A \geq [v \ v \ \ldots \ v],$$

$$p_1 + \cdots + p_m \leq 1, \quad v, p_1, p_2, \ldots, p_m \geq 0.$$

For example,

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad \text{gives} \quad v \leq p_1 + 3p_2, \quad v \leq 2p_1 + 4p_2, \quad \text{etc.}$$

If A is $m \times n$ and $n < m$, then we know that there is an optimum strategy where at most n of $p_1, \ldots, p_m$ are nonzero.

If A is a 2×2 matrix, then either (1) the duality gap is zero, or (2) Alice and Betty have mixed strategies where the values are balanced, e.g.,

$$\vec{p}^T A = [v \ v]$$

for Alice.

For any stochastic $\vec{p}$ and $\vec{q}$ we have

$$\text{MinEntry}(\vec{p}^{\mathrm{T}} A) \leq \text{MaxEntry}(A\vec{q})$$

and if these are equal then this common value is the value of (the mixed strategy games of A).

LP standard form: maximize $\vec{c} \cdot \vec{x}$, subject to $A\vec{x} \leq \vec{b}$, $\vec{x} \geq \vec{0}$.

Unbounded LP: A variable enters, but nothing leaves.

2-phase method: (1) introduce x_0 on right, (2) pivot x_0 into the basis for a feasible dictionary, and try to maximize $w = -x_0$, (3) if w reaches 0, pivot x_0 out of dictionary and eliminate all x_0; e.g.,

$$\begin{aligned} x_4 &= \ -7 + \cdots + x_0 \\ x_9 &= \ -8 + \cdots + x_0 \end{aligned} \qquad x_0 \text{ enters, } x_9 \text{ leaves}$$

Degenerate pivots: say x_5 enters, and have $x_3 = 0 + x_2 - 2x_5 + \cdots$ Then x_3 cannot tolerate any positive x_5 value, and leaves without changing the basic feasible solution (and z value). Degenerate pivots not necessarily bad, but cycling can only occur when all pivots in the cycle are degenerate.

Perturbation method: Add ϵ_1 to first inequality, ϵ_2 to second inequality, etc., $1 >> \epsilon_1 >> \epsilon_2 >> \cdots$. Never has a degenerate pivot (wrt the ϵ_i's), since dictionary pivots represent invertible linear transformations (which can't have a row of zeros). In more detail, we have

$$\vec{x}_B = A_B^{-1}(\vec{b} + \vec{\epsilon} - A_N\vec{x}_N)$$

and since A_B is the inverse of a matrix, it cannot have a row of all 0's, and hence each entry of $A_B^{-1}\vec{\epsilon}$ is nonzero.

The formulas for simplex method dictionaries (in standard form) is

$$\begin{aligned} \vec{x}_B &= A_B^{-1}\vec{b} - A_B^{-1}A_N\vec{x}_N \\ \zeta &= \vec{c}_B^{\mathrm{T}}A_B^{-1}\vec{b} + (\vec{c}_N^{\mathrm{T}} - \vec{c}_B^{\mathrm{T}}A_B^{-1}A_N)\vec{x}_N \end{aligned}$$

In the computation above, we compute $\vec{c}_B^{\mathrm{T}}A_B^{-1}A_N$ by first computing $\vec{c}_B^{\mathrm{T}}A_B^{-1}$, and then multiplying the result (a row vector) times A_N; it would be more expensive to first compute $A_B^{-1}A_N$.

For the A_B^{-1} of the $i-1$-th and i-th dictionaries we have

$$A_{B_i}^{-1} = E_i A_{B_{i-1}}^{-1}$$

where E_i is an eta matrix, equal to the identity except in one column. This formula can be applied recursively to get

$$A_{B_{i+k}}^{-1} = E_{i+k}E_{i+k-1}\cdots E_i A_{B_{i-1}}^{-1};$$

it turns out that due to the cost in FLOPS, the eta it is best to use k up to roughly between $\sqrt{m}$ and m (if there are m basic variables); there are also roundoff error issues that are not analyzed in Vanderbei.

For any basis, B, A_B must be invertible, and hence there can be no linear dependence between rows of A_B (or between its columns).

Let the b-th row in a matrix game be $\vec{f}(b)$. If $\vec{f}$ is a convex function (i.e., concave up), then Alice has an optimal strategy that is some combination of the smallest and largest values of b (i.e., the top and bottom rows). If $\vec{f}$ is concave down, then Alice has an optimal strategy this is some combination of two adjacent rows. (These combinations can be 100% of one row in certain cases.)

The dual to (1) maximize $\vec{c}\cdot\vec{x}$ subject to $A\vec{x} \le \vec{b}$ and $\vec{x} \ge \vec{0}$ is (2) maximize $-\vec{b}\cdot\vec{y}$ subject to $A^{\mathrm{T}}\vec{y} \ge \vec{c}$ and $\vec{y} \ge 0$. If both these LP's are feasible, then for $\vec{x}, \vec{y}$ feasible the following are equivalent: (1) $\vec{x}, \vec{y}$ are optimal solutions; (2) $\vec{c}\cdot\vec{x} = \vec{b}\cdot\vec{y}$ (Strong Duality Theorem); (3) $x_i z_i = 0$ for all i and $y_j w_j = 0$ for all j, where the z_i's are the dual slack variables and the w_j's are the primal slack variables (Complementary Slackness). Furthermore, for any $\vec{x}, \vec{y}$ feasible we have $\vec{c}\cdot\vec{x} \le \vec{b}\cdot\vec{y}$ (Weak Duality Theorem).