Marks

[6] **1.** Consider the two matrix games

$$A_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

(a) Assume that A_1 is irreducible (i.e., that every strategy is essential) and use linear algebra to compute (or attempt to compute) the value of the game and the equilibrium strategies of both players. Based on your computations, was the irreducibility assumption on A_1 was correct? Explain carefully.

(b) Same question for A_2.

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[6] **2.** A chef prepares tofu and cookies, and will pair them with teriyaki sauce and melted chocolate. Her profits are as follows: chocolate tofu: 100; teriyaki tofu: 75; chocolate cookies: 110; teriyaki cookies: 125.

(a) Write this matching problem as a linear program.

(b) Say that we "price" chocolate at 100 and teriyaki at 115; carefully explain what upper bound this gives on this linear program. Does this give the true optimal?

(c) Same as (b) for pricing chocolate at 100 and teriyaki at 50.

(d) Write the dual LP, and explain how any pricing of the chocolate and teriyaki leads to a feasible solution to the dual. Illustrate this with the two pricings in parts (b) and (c).

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[6] **3.** Consider the problem: maximize $x_1 + x_2$ subject to $2x_1 + x_2 \leq 3$, $x_1 + 3x_2 \leq 4$, and $x_1, x_2 \geq 0$. Write the slack variables for this linear program, and write down the dual linear program and dual slack variables.

Check to see if the following are optimal solutions to the primal linear program using complementary slackness:

(a) $x_1 = 0$, $x_2 = 1$;

(b) $x_1 = 3/2$, $x_2 = 0$;

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[6] **4.** Solve the following two linear progams using the two-phase method, **adding an auxilliary variable x_0 to EVERY slack variable equation in the dictionary:**

(a) maximize x_1 subject to $x_1 + x_2 \leq 5$, $x_1 \geq 6$, $x_1, x_2 \geq 0$.

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(b) maximize x_1 subject to $x_1 + x_2 \leq 5$, $x_1 \geq 2$, $x_1, x_2 \geq 0$.

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[6] **5.** Say you run the revised simplex algorithm with algorithms that (i) invert an $m \times m$ matrix, A, in time m^3, and (ii) apply this inverse to a vector (either on the left or right) in time m^2.

(a) Explain carefully how much time you spend on the i-th pivot after inverting A_B.

(b) Assuming you run a revised simplex algorithm that inverts A_B every t pivots, what is your average computation time per pivot?

(c) Based only on time considerations, how often should we invert the matrix A_B in the revised simplex algorithm?

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[6] **6.** Consider the problem

$$\text{maximize} \quad 5x_1 + 6x_2 \quad \text{subject to} \quad x_1, x_2 \geq 0,$$

$$\begin{array}{lll} x_1 & +2x_2 & \leq 5, \\ x_1 & +x_2 & \leq t, \end{array}$$

where t is a real parameter. This optimal dictionary and B^{-1} (as in the revised simplex method) for this problem when $t = 3$ is

$$\begin{array}{lllll} x_1 & = 1 & +x_3 & -2x_4, \\ x_2 & = 2 & -x_3 & +x_4, \\ z & = 17 & -x_3 & -4x_4, \end{array} \qquad B^{-1} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

(a) Just by looking at the original linear program, what happens when $t < 0$? Explain. What happens with t is small but positive? Explain.

(b) Write the above optimal dictionary for general t. For what range of values of t is this dictionary still optimal?

(c) Say t is a little bigger than 5. Make one dual pivot to find a few optimal dictionary. For what t is this optimal?

(d) Solve the above LP for all real values of t, and make a plot of $z^*(t)$.

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[6] **7.** Let $f(x, y)$ be a function in two real variables, x, y. Say that f is *x-concave down* if for any $x_1, x_2, y \in \mathbf{R}$ we have

$$f\big((x_1 + x_2)/2, y\big) \geq (1/2)f(x_1, y) + (1/2)f(x_2, y).$$

(a) Let $F_1(x) = \min_y f(x, y)$ (assume that for each x this minimum is achieved for some value of y). Show that if f is *x-concave down* then for any $x_1, x_2 \in \mathbf{R}$ we have

$$F_1\big((x_1 + x_2)/2\big) \geq (1/2)F_1(x_1) + (1/2)F_1(x_2).$$

(b) Define what it should mean for f to be *y-convex* so that we have an analogous condition for $F_2(y) = \max_x f(x, y)$.

(c) Explain how the ideas in (a) and (b) relate to matrix games; use pure strategies in Rock-Paper-Scissors as an example.

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The University of British Columbia

Final Examinations - April 2008

Mathematics 340–201

Closed book examination

Time: 2.5 hours

Name ____________________ **Signature** ____________________

Student Number____________________ **Instructor's Name** ____________________

Section Number ____________________

Special Instructions:

Calculators, notes, or other aids may not be used. Answer questions on the exam. Note that the exam is two-sided! A note sheet is provided with the exam. You have two extra pages at the back for additional space.

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