Marks

[15] **1.**

(a) Consider the problem

$$\text{maximize} \quad -x_2 \quad \text{subject to} \quad x_1, x_2 \geq 0,$$

$$\begin{array}{rrl} -x_1 & -x_2 & \leq -2, \\ x_1 & -x_2 & \leq 0. \end{array}$$

Use the two-phase method to solve this problem, with the largest coefficient rule to choose the entering variable, breaking ties by choosing the smallest subscript. Clearly state which variable enters and which leaves on each pivot, and what your final dictionary means in terms of the original problem.

(b) Check your work to part (a), by taking the z-row of your final dictionary and replacing z and the slack variables by expressions involving the decision variables (which can be obtained from the initial dictionary); the two sides of the equation should agree.

(c) Consider the problem

$$\text{maximize} \quad x_1 + x_2 \quad \text{subject to} \quad x_1, x_2 \geq 0,$$

$$\begin{array}{rrl} -x_1 & -x_2 & \leq -2, \\ x_1 & -x_2 & \leq 0. \end{array}$$

Use the two-phase method to solve this problem and check your work in the same fashion as was done for the first LP in parts (a) and (b).

[15] **2.** Consider the problem of minimizing $-x_1 + ax_2$ subject to $x_1^2 + x_2^2 \leq 2$ and $x_1 \leq 1$, where a is a real parameter. For what values of a is $(x_1, x_2) = (1, 1)$ a KKT point?

[15] **3.** Consider our favourite LP and its corresponding final dictionary:

$$\begin{array}{lrrl} \text{max.} & 4x_1 + 5x_2 & \text{s. t.} & x_1, x_2 \geq 0, \\ & x_1 & +x_2 & \leq 5, \\ & x_1 & +2x_2 & \leq 8, \\ & 2x_1 & +x_2 & \leq 8. \end{array} \qquad \begin{array}{lcrrr} x_1 & = & 2 & -2x_3 & +x_4 \\ x_2 & = & 3 & +x_3 & -x_4 \\ x_5 & = & 1 & +3x_3 & -x_4 \\ z & = & 23 & -3x_3 & -x_4 \end{array}$$

Write down the dual LP. Make a one-to-one correspondence between primal variables and dual variables. Write down the dictionary in the dual LP that is dual to the above final (primal) dictionary. Write the primal LP as a minimization of $f(\mathbf{x})$ subject to $\mathbf{g}(\mathbf{x}) \leq 0$; what are f and $\mathbf{g}$? Verify that $(2, 3)$ is a KKT point. How is this KKT verification related to the optimal solution of the dual LP?

[35] **4.** A candy bar has 150mg of caffeine, 40g of sugar, and costs \$1. A can of soda has 50mg of caffeine, 0g of sugar, and costs \$1. A cappuccino has 150mg of caffeine, 20g

Continued on page 3

of sugar, and costs \$2. You wish to maximize the caffeine supplied by a combination of these three foods while getting at most 100g of sugar and spending at most \$7. You are lead to the LP

$$\text{maximize} \quad 150x_1 + 50x_2 + 150x_3 \quad \text{subject to} \quad x_1, x_2, x_3 \geq 0,$$

$$\begin{array}{llll} 40x_1 & & +20x_3 & \leq 100, \\ x_1 & +x_2 & +2x_3 & \leq 7. \end{array}$$

You solve this LP, with a final dictionary that reads

$$\begin{array}{rclllll} x_1 & = & 1 & +\frac{1}{3}x_2 & -\frac{1}{30}x_4 & +\frac{1}{3}x_5 \\ x_3 & = & 3 & -\frac{2}{3}x_2 & +\frac{1}{60}x_4 & -\frac{2}{3}x_5 \\ z & = & 600 & & -\frac{5}{2}x_4 & -50x_5 \end{array}$$

(a) Write out B and B^{-1} for the final dictionary, and label the rows and columns of these matrices with the appropriate units.

(b) Interpret the top left entry of B^{-1} and say in what units (e.g. feet per second) this entry is; if we were willing to spend \$8 instead of \$7 (i.e. one additional dollar), what do the B^{-1} entries tell you about the new x_1, x_3 rows for the adjusted final dictionary?

(c) The can of soda is modified to contain 60mg of caffeine (everything else remains the same). Derive an adjusted dictionary from the final dictionary, and make one pivot to find the new optimal solution.

(d) Consider the original problem (where a can of soda has 50mg of caffeine), but where the money we are willing to spend is b_2, a parameter. Use $B^{-1}b$ and $c_B B^{-1}b$ to find the dictionary with x_1, x_3 basis (as in the final dictionary), but where b_2 is a parameter. Check you work by checking that $b_2 = 7$ gives you the above final dictionary.

(e) For all non-negative values of b_2 find the maximum caffeine you can achieve and what combinations of foods are needed to achieve that maximum. Give a simple explaination for why your solution is optimal when $b_2 = .0001$ (in terms of the original setting, e.g. candy bars, soda, sugar, etc.).

(f) Consider the original problem (where we spend at most \$7), but where a can of soda has t grams of sugar, where t is a non-negative real parameter. Consider the dictionary with x_1, x_3 basic corresponding to this problem. Use the formulas at the end of this exam to say which parts of the dictionary (i.e. coefficients and constants) will (or may) involve t; give a formula for all such parts. For which values of t is this dictionary optimal? Give a simple explanation for why this is true (in terms of the original setting).

[20] **5.** Consider the problem,

$$\text{maximize} \quad 3x_1 + 2x_2 + 4x_3 \quad \text{subject to} \quad x_1, x_2, x_3 \geq 0,$$

$$\begin{array}{llll} x_1 & +x_2 & +2x_3 & \leq 4, \\ 2x_1 & & +3x_3 & \leq 5, \\ 2x_1 & +x_2 & +3x_3 & \leq 7. \end{array}$$

Continued on page 4

Use the revised simplex method with the smallest subscript rule (e.g. the entering variable is the one with the smallest subscript among those whose coefficient in the z row is positive) to solve this LP.

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Formulae

The following formulae may be of use. They will not be explained; you are assumed to understand what they mean and to what they refer.

$$\begin{array}{rll} x_B = & B^{-1}b & -B^{-1}A_N x_N \\ z = & c_B^T B^{-1} b & +(c_N^T - c_B^T B^{-1} A_N) x_N \end{array}$$

$$y^T = c_B^T B^{-1}, \qquad y^T B = c_B^T, \qquad d = B^{-1}a, \qquad Bd = a, \qquad x_B^* - td$$

$$B_{i+1} = B_i E_{i+1}, \qquad \text{with } E_{i+1} \text{ formed using } d \text{ from } i\text{-th dictionary}$$

$$\min f(x), \text{ s.t. } g(x) \le 0, \qquad \nabla f + u_1 \nabla g_1 + \cdots + u_n \nabla g_n = 0$$

$$u_i \ge 0, \qquad u_i = 0 \text{ if } g_i \text{ is inactive.}$$

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The University of British Columbia

Final Examinations - December 2000

Mathematics 340–102

Closed book examination

Time: $2\frac{1}{2}$ hours

Name ______________________________ **Signature** ______________________________

Student Number ___________________ **Instructor's Name** ______________________

Section Number ______________________

Special Instructions:

Candidates may not use any notes or calculators. A list of formulae is provided at the end of this exam. Answer questions in the booklets provided.

Rules governing examinations

1. Each candidate should be prepared to produce his library/AMS card upon request.

2. Read and observe the following rules:

No candidate shall be permitted to enter the examination room after the expiration of one half hour, or to leave during the first half hour of the examination. Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.

CAUTION - Candidates guilty of any of the following or similar practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.

(a) Making use of any books, papers or memoranda, other than those authorized by the examiners.

(b) Speaking or communicating with other candidates.

(c) Purposely exposing written papers to the view of other candidates. The plea of accident or forgetfulness shall not be received.

3. Smoking is not permitted during examinations.

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Total		100