Math 340 Note Sheet for Midterm, Fall 2014

$$\text{ValueAliceAnnouncesPure}(A) = \max_i \text{MinEntry of } i\text{-th row of } A = \max_i \min_j a_{ij}$$

$$\text{ValueBettyAliceAnnouncesPure}(A) = \min_j \text{MinEntry of } j\text{-th column of } A = \min_j \max_i a_{ij}$$

$$\text{DualityGap} = (\text{ValueBettyAnnouncesPure}) - (\text{ValueAliceAnnouncesPure})$$

The value of Alice announces a mixed strategy is

$$\min_{\vec{p} \text{ stoch}} \text{MinEntry}(\vec{p}^{\mathrm{T}} A)$$

Is given by LP

$$\max v \quad \text{s.t.} \quad \vec{p}^T A \geq [v\ v\ \dots\ v],$$

$$p_1 + \cdots + p_m = 1, \quad p_1, p_2, \dots, p_m \geq 0.$$

If all entries of A are positive this is equivalent to

$$\max v \quad \text{s.t.} \quad \vec{p}^T A \geq [v\ v\ \dots\ v],$$

$$p_1 + \cdots + p_m \leq 1, \quad v, p_1, p_2, \dots, p_m \geq 0.$$

For example,

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad \text{gives} \quad v \leq p_1 + 3p_2, \quad v \leq 2p_1 + 4p_2, \quad \text{etc.}$$

If A is $m \times n$ and $n < m$, then we know that there is an optimum strategy where at most n of $p_1, \dots, p_m$ are zero.

If A is a 2×2 matrix, then either (1) the duality gap is zero, or (2) Alice and Betty have mixed strategies where the values are balanced, e.g.,

$$\vec{p}^T A = [v\ v]$$

for Alice.

For any stochastic $\vec{p}$ and $\vec{q}$ we have

$$\text{MinEntry}(\vec{p}^{\mathrm{T}} A) \leq \text{MaxEntry}(A\vec{q})$$

and if these are equal then this common value is the value of (the mixed strategy games of A).

LP standard form: maximize $\vec{c} \cdot \vec{x}$, subject to $A\vec{x} \leq \vec{b}$, $\vec{x} \geq \vec{0}$.

Unbounded LP: A variable enters, but nothing leaves.

2-phase method: (1) introduce x_0 on right, (2) pivot x_0 into the basis for a feasible dictionary, and try to maximize $w = -x_0$, (3) if w reaches 0, pivot x_0 out of dictionary and eliminate all x_0; e.g.,

$$\begin{aligned} x_4 &= \ -7 + \cdots + x_0 \\ x_9 &= \ -8 + \cdots + x_0 \end{aligned} \qquad x_0 \text{ enters, } x_9 \text{ leaves}$$

Degenerate pivots: say x_5 enters, and have $x_3 = 0 + x_2 - 2x_5 + \cdots$ Then x_3 cannot tolerate any positive x_5 value, and leaves without changing the basic feasible solution (and z value). Degenerate pivots not necessarily bad, but cycling can only occur when all pivots in the cycle are degenerate.

Perturbation method: Add ϵ_1 to first inequality, ϵ_2 to second inequality, etc., $1 >> \epsilon_1 >> \epsilon_2 >> \cdots$. Never has a degenerate pivot (wrt the ϵ_i's), since dictionary pivots represent invertible linear transformations (which can't have a row of zeros). In more detail, we have

$$\vec{x}_B = A_B^{-1}(\vec{b} + \vec{\epsilon} - A_N \vec{x}_N)$$

and since A_B is the inverse of a matrix, it cannot have a row of all 0's, and hence each entry of $A_B^{-1}\vec{\epsilon}$ is nonzero.