**Math 340 Note Sheet for the Final Exam, Fall 2014**

$$\text{ValueAliceAnnouncesPure}(A) = \max_i \text{MinEntry of } i\text{-th row of } A = \max_i \min_j a_{ij}$$

$$\text{ValueBettyAliceAnnouncesPure}(A) = \min_j \text{MaxEntry of } j\text{-th column of } A = \min_j \max_i a_{ij}$$

$$\text{DualityGap} = (\text{ValueBettyAnnouncesPure}) - (\text{ValueAliceAnnouncesPure})$$

The value of Alice announces a mixed strategy is

$$\max_{\vec{p}\ \text{stoch}} \text{MinEntry}(\vec{p}^{\mathrm{T}} A)$$

Is given by LP

$$\max v \quad \text{s.t.} \quad \vec{p}^T A \geq [v\ v\ \ldots\ v],$$

$$p_1 + \cdots + p_m = 1, \quad p_1, p_2, \ldots, p_m \geq 0.$$

If all entries of $A$ are positive this is equivalent to

$$\max v \quad \text{s.t.} \quad \vec{p}^T A \geq [v\ v\ \ldots\ v],$$

$$p_1 + \cdots + p_m \leq 1, \quad v, p_1, p_2, \ldots, p_m \geq 0.$$

For example,

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad \text{gives} \quad v \leq p_1 + 3p_2, \quad v \leq 2p_1 + 4p_2, \quad \text{etc.}$$

If $A$ is $m \times n$ and $n < m$, then we know that there is an optimum strategy where at most $n$ of $p_1, \ldots, p_m$ are nonzero.

If $A$ is a $2 \times 2$ matrix, then either (1) the duality gap is zero, or (2) Alice and Betty have mixed strategies where the values are balanced, e.g.,

$$\vec{p}^T A = [v\ v]$$

for Alice.

For any stochastic $\vec{p}$ and $\vec{q}$ we have

$$\text{MinEntry}(\vec{p}^{\mathrm{T}} A) \leq \text{MaxEntry}(A\vec{q})$$

and if these are equal then this common value is the value of (the mixed strategy games of $A$).

LP standard form: maximize $\vec{c} \cdot \vec{x}$, subject to $A\vec{x} \leq \vec{b}$, $\vec{x} \geq \vec{0}$.

Unbounded LP: A variable enters, but nothing leaves.

2-phase method: (1) introduce $x_0$ on right, (2) pivot $x_0$ into the basis for a feasible dictionary, and try to maximize $w = -x_0$, (3) if $w$ reaches 0, pivot $x_0$ out of dictionary and eliminate all $x_0$; e.g.,

$$\begin{aligned} x_4 &= \ -7 + \cdots + x_0 \\ x_9 &= \ -8 + \cdots + x_0 \end{aligned} \qquad x_0 \text{ enters, } x_9 \text{ leaves}$$

Degenerate pivots: say $x_5$ enters, and have $x_3 = 0 + x_2 - 2x_5 + \cdots$ Then $x_3$ cannot tolerate any positive $x_5$ value, and leaves without changing the basic feasible solution (and $z$ value). Degenerate pivots not necessarily bad, but cycling can only occur when all pivots in the cycle are degenerate.

Perturbation method: Add $\epsilon_1$ to first inequality, $\epsilon_2$ to second inequality, etc., $1 >> \epsilon_1 >> \epsilon_2 >> \cdots$. Never has a degenerate pivot (wrt the $\epsilon_i$'s), since dictionary pivots represent invertible linear transformations (which can't have a row of zeros). In more detail, we have

$$\vec{x}_B = A_B^{-1}(\vec{b} + \vec{\epsilon} - A_N \vec{x}_N)$$

and since $A_B$ is the inverse of a matrix, it cannot have a row of all 0's, and hence each entry of $A_B^{-1}\vec{\epsilon}$ is nonzero.

The formulas for simplex method dictionaries (in standard form) is

$$\vec{x}_B = A_B^{-1}\vec{b} - A_B^{-1}A_N\vec{x}_N$$
$$\zeta = \vec{c}_B^{\mathrm{T}}A_B^{-1}\vec{b} + (\vec{c}_N^{\mathrm{T}} - \vec{c}_B^{\mathrm{T}}A_B^{-1}A_N)\vec{x}_N$$

In the computation above, we compute $\vec{c}_B^{\mathrm{T}}A_B^{-1}A_N$ by first computing $\vec{c}_B^{\mathrm{T}}A_B^{-1}$, and then multiplying the result (a row vector) times $A_N$; it would be more expensive to first compute $A_B^{-1}A_N$.

For the $A_B^{-1}$ of the $i-1$-th and $i$-th dictionaries we have

$$A_{B_i}^{-1} = E_i A_{B_{i-1}}^{-1}$$

where $E_i$ is an eta matrix, equal to the identity except in one column. This formula can be applied recursively to get

$$A_{B_{i+k}}^{-1} = E_{i+k}E_{i+k-1}\cdots E_i A_{B_{i-1}}^{-1};$$

it turns out that due to the cost in FLOPS, the eta it is best to use $k$ up to roughly between $\sqrt{m}$ and $m$ (if there are $m$ basic variables); there are also roundoff error issues that are not analyzed in Vanderbei.

For any basis, $B$, $A_B$ must be invertible, and hence there can be no linear dependence between rows of $A_B$ (or between its columns).

Let the $b$-th row in a matrix game be $\vec{f}(b)$. If $\vec{f}$ is a convex function (i.e., concave up), then Alice has an optimal strategy that is some combination of the smallest and largest values of $b$ (i.e., the top and bottom rows). If $\vec{f}$ is concave down, then Alice has an optimal strategy this is some combination of two adjacent rows. (These combinations can be 100% of one row in certain cases.)

The dual to (1) maximize $\vec{c}\cdot\vec{x}$ subject to $A\vec{x} \le \vec{b}$ and $\vec{x} \ge \vec{0}$ is (2) maximize $-\vec{b}\cdot\vec{y}$ subject to $A^{\mathrm{T}}\vec{y} \ge \vec{c}$ and $\vec{y} \ge 0$. If both these LP's are feasible, then for $\vec{x}, \vec{y}$ feasible the following are equivalent: (1) $\vec{x}, \vec{y}$ are optimal solutions; (2) $\vec{c}\cdot\vec{x} = \vec{b}\cdot\vec{y}$ (Strong Duality Theorem); (3) $x_i z_i = 0$ for all $i$ and $y_j w_j = 0$ for all $j$, where the $z_i$'s are the dual slack variables and the $w_j$'s are the primal slack variables (Complementary Slackness). Furthermore, for any $\vec{x}, \vec{y}$ feasible we have $\vec{c}\cdot\vec{x} \le \vec{b}\cdot\vec{y}$ (Weak Duality Theorem).