Marks

[10] **1.** Consider the problem: maximize x_1 subject to $x_1 \geq 4$, $x_1 \geq 8$, $x_1 \geq 0$. Write this as a linear program in standard form. Use the two-phase method, **adding an auxilliary variable x_0 to EVERY slack variable equation in the dictionary,** to solve this LP. Use the smallest subscript rule to break any ties for entering or leaving variables.

Answer: In standard form this is

$$\max x_1, \quad \text{subject to}$$
$$-x_1 \leq -4$$
$$-x_1 \leq -8$$
$$x_1 \geq 0$$

This gives rise to an initial dictionary:

$$\zeta = x_1$$
$$w_1 = -4 + x_1$$
$$w_2 = -8 + x_1$$

Due to the negative constants in the constraints, we use the two phase method, beginning by adding x_0 as non-basic and modifying the objective:

$$\zeta = -x_0$$
$$w_1 = -4 + x_1 + x_0$$
$$w_2 = -8 + x_1 + x_0$$

We have x_0 enter and choose the leaving variable so that the dictionary becomes feasible, namely w_2 leaves:

$$\zeta = -8 + x_1 - w_2$$
$$w_1 = 4 + \quad + w_2$$
$$x_0 = 8 - x_1 + w_2$$

So x_1 enters and x_0 leaves:

$$\zeta = x_0$$
$$w_1 = 4 + \quad + w_2$$
$$x_1 = 8 - x_0 + w_2$$

This gives us a feasible dictionary with x_0 non-basic. So we eliminate the x_0 to get the feasible dictionary (and use our original objective $\zeta = x_1$), to start the second phase:

$$\zeta = 8 + w_2$$
$$w_1 = 4 + w_2$$
$$x_1 = 8 + w_2$$

So w_2 enters and nothing leaves (i.e., w_2 is unconstrained), so w_2, and hence ζ, can increase without bound. Hence this linear program is unbounded.

Continued on page 3

[20] **2.** (4 points for each part) Briefly justify your answers:

(a) Find the value of "Alice announces a pure strategy" and "Betty announces a pure strategy" for the matrix game:

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$$

What do you conclude about the value of the mixed strategy games?

Answer: If Alice chooses the top row, Betty picks column 1, for a payout of 1; if Alice chooses the bottom row, Betty picks column 2, for a payout of 3. Hence Alice picks row 2, and the value of "Alice announces a pure strategy" is 3.

Similarly Betty picks column two so that Alice gets a payout of 3 in "Betty announces a pure strategy" (if Betty picks column one then Alice gets a payout of 4).

Since the duality gap is 0, the value of the mixed strategy games is also 3.

(b) In the matrix game

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 4 & 3 & 1 \\ 5 & 2 & 1 \end{bmatrix}$$

someone claims that Alice's optimal mixed strategy is $[1/2\ 1/2\ \ 0]$ and that Betty's is $[1/3\ 1/3\ 1/3]$. Make a quick calculation (without using the simplex method) to determine whether or not this is true; explain why your quick calculation works.

Answer: We have

$$[1/2\ 1/2\ \ 0]A = [5/2\ 5/2\ 5/2],$$

whose minimum entry is $5/2$. Also

$$A \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 7/3 \\ 8/3 \\ 8/3 \end{bmatrix},$$

whose maximum entry is $8/3$.

So when Alice plays the mixed strategy $[1/2\ 1/2\ \ 0]$, Alice's payout is $5/2$, and when Betty plays $[1/3\ 1/3\ 1/3]$, the payout to Alice is $8/3$. Since $5/2$ and $8/3$ are different, we are not at optimality (at least one strategy is not optimal).

Continued on page 4

Some people made conclusions about Alice's strategy being optimal because $[1/2\ 1/2\ \ 0]A$ **is balanced. This is not necessarily true. For example, in part (a) we have that Alice's optimal strategy is** $[0\ 1]$**, and yet**

$$[0\ 1]\begin{bmatrix}1 & 2\\ 4 & 3\end{bmatrix} = [4\ 3],$$

which is not balanced. Also you should be careful to note that what is true for 2×2 **matrix games is not always true for larger games; for example, a** 3×3 **matrix game can be reducible without having one row (or column) dominated by another.**

Continued on page 5

(c) Let A be a 100×2 matrix game. Argue that Alice has an optimal mixed strategy where she plays at most two rows.

Answer: By adding a sufficiently large positive integer to each entry of A, we get an equivalent game where A has all strictly positive entries. In this case the linear program for Alice's optimal strategy, $\vec{p}$, will be to maximize v (which will be strictly positive), subject to the three constraints

$$\vec{p}^{\mathrm{T}} A \geq [v \ v], \quad p_1 + \cdots + p_{100} \leq 1$$

[the fact that it suffices to write ≤ 1 was a homework problem and explained in class, and is due to the fact that A's entries are strictly positive, so to maximize v, the vector $\vec{p}$ will never be taken to have a sum less than 1].

It follows that we can run the simplex on the above linear program, which has three basic variables. At the final dictionary, v will have a strictly positive value, so v will be in the basis as well as at most two of the p_i. Hence at least 98 of the p_i will be non-basic, and hence equal to zero at optimality.

(d) and (e)

Answer: There are a number of possible answers; for part (d) you have to make sure that all possible entering variables give a degenerate pivot; part (e) requires you to show a second degenerate pivot, which cannot be determined from the initial dictionary without some argument.

One type example is to maximize x_1 subject to $x_1 \leq x_2$ and $x_2 \leq x_3$ (and $x_1, x_2, x_3 \geq 0$). The dictionary looks like

$$\zeta = x_1, \quad w_1 = x_2 - x_1, \quad w_2 = x_3 - x_2.$$

So x_1 must enter, and hence w_1 leaves, a degenerate pivot. The next dictionary is

$$\zeta = x_2 - w_1, \quad x_1 = x_2 - w_1, \quad w_2 = x_3 - x_2,$$

and we note that the last line remains the same (regardless of the pivot). So x_2 must enter, and w_2 must leave, another degenerate pivot. Note that if we replace the objective $\zeta = x_1$ with $\zeta = x_1 + x_2 + x_3$ (or something like that), then some possible first pivots are not degenerate.

A variant of the above is something like

$$\zeta = x_1 + x_2, \quad w_1 = -x_1, \quad w_2 = -x_2.$$

Continued on page 6

Here either x_1 or x_2 can enter, but the slack variables w_1 and w_2 work independently on x_1 and x_2. So if x_1 enters, then w_1 leaves, giving a new dictionary

$$\zeta = -w_1 + x_2, \quad x_1 = -w_1, \quad w_2 = -x_2.$$

Then x_2 enters with a degenerate pivot. On the other hand, if x_2 enters first then we get the dictionary

$$\zeta = x_1 - w_2, \quad w_1 = -x_1, \quad x_2 = -w_2,$$

and again the second pivot (x_1 enters, w_1 leaves) is degenerate.

Another type of example comes from game theory linear programs, such as

$$\zeta = v, \quad w_1 = -v + p_1 + 2p_2, \quad w_2 = -v + etc.$$

We easily see that the first pivot, with v entering is degenerate. The second pivot, however, will usually have a possible degenerate pivot, but there will usually be a possible non-degenerate pivot unless there are extra constraints (such as in some 2×3 or $2\times n$ with $n \geq 3$) (or unless our hand is forced by using the perturbation method, but then the pivot isn't really degenerate).

A lot of people assumed that if the objective was $x_1 + x_2$, then the entering variables in the first two pivots would necessarily be x_1 and x_2; this is not true.

Additional remarks:

1. **For the final exam, you definitely want know the basic algorithms used in this course, including simplex method (including two-phase method and perturbation method) and how to solve a 2×2 matrix game. You should iron out any problems regarding unbounded dictionaries and when to stop the simplex method (a number of people kept going on Problem 1 beyond the w_2 entering the basis unboundedly).**
2. **Getting a 4 out of 4 on a problem does not mean you have done it perfectly. For example, on Problem 2, part (c), a lot of people tacitly assumed that A has all positive entries; in general you have to modify A in order to make this assumption. Please look at the solutions; although I ignored a number of such details on the midterm, I hope to see a more complete answers on the final exam.**
3. **The solutions given here are very long and detailed; such detail is not needed for your solution; for example, on Problem 1, you don't have to write "due to the negative constraints etc." You could write "infeasible dictionary, so add x_0's."**

The End

Be sure that this examination has 6 pages including this cover

The University of British Columbia

Midterm Examinations - October 2014

Mathematics 340

Closed book examination Time: 50 minutes

Name ______________________________ **Signature** ______________________________

Student Number__________________ **Instructor's Name** __________________

Section Number __________________

Special Instructions:

Calculators, notes, or other aids may not be used. Answer questions on the exam. This exam is two-sided!

Rules governing examinations

1		10
2		20
Total		30