

# MATH 340: MATRIX GAMES AND POKER

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## CONTENTS

1. Matrix Games	2
1.1. A Poker Game	3
1.2. A Simple Matrix Game: Rock/Paper/Scissors	3
1.3. A Simpler Game: Even/Odd Pennies	3
1.4. Some Examples	4
2. The Values of a Matrix Game	4
2.1. Alice Announces a Pure Strategy	4
2.2. Betty Announces a Pure Strategy	4
2.3. <b>Maxmin</b> $\leq$ <b>Minmax</b>	5
2.4. Alice Announces a Mixed Strategy	5
2.5. Betty Announces a Mixed Strategy	5
2.6. The Duality Theorem for Matrix Games	6
2.7. The Definition of the (Mixed-Strategies) Value of a Matrix Game	6
2.8. Summary of this Section	7
3. More on Poker Games	7
3.1. Everywhere Better Rows and Columns	8
3.2. The Payout Matrix in Poker Games	8
4. Remarkable Theorems	10
5. Two By Two Matrix Games	11
6. Terminology	13
7. The Duality Gap	13
8. THE BIG POKER QUESTION: A $2^{52} \times 2$ Poker Game	14
9. Learning Goals and Sample Exam Problems	15
Appendix A. Exercises Beyond Exam Questions	20
Appendix B. Review of Matrix Notation	21

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Math 340 will begin by describing the idea of a “matrix game” and making some preliminary remarks about such games. These preliminary remarks appear here. We will also refer to *Two-Person Zero-Sum Games* by Alan Washburn [Was14].

We then follow Vanderbei’s text [Van14] to learn some fundamental aspects of linear programming. We will then revisit matrix games.

The plan is to cover Sections 1–6 in roughly the first two weeks of classes, and then to revisit these sections in the last few weeks of classes, and to discuss Section 7 then.

Section 6 is short but very important, since it describes terminology that was used in earlier versions of the course and/or terminology that is commonly used in the literature; this will be helpful when looking at previous homework questions, and in searching the internet (if you wish to do so) to get more background and examples regarding matrix games.

The goals of Sections 1–5 are to (1) introduce you to some aspects of matrix games, and (2) to show you some interesting examples, such as why “bluffing” can be a necessary part of a strategy in poker. We will need to study the foundations of linear programming to understand how to find “optimal strategies” for such games. We warn the reader that all the commonly played forms of Poker involve very large matrices (i.e., a large number of “pure” strategies), and we will only be able to completely analyze some very simple types of Poker games (really “toy” Poker games) in this article.

## 1. MATRIX GAMES

Matrix games can be viewed as a very special case of linear programming. They include tons of common two person games (from poker, rock/paper/scissors, matching pennies, battleship, etc.). These games are fun, yet they illustrate loads of general principles of linear programming (e.g., duality, dominance of strategies, symmetry, deviation).

In our discussion, there will be a default naming convention: “Player A” named “Alice,” and the rows of the matrix represent Alice’s pure strategies; Alice plays against a “Player B” named “Betty” whose pure strategies are represented by the columns.

1.1. **A Poker Game.** Consider the following game:

Step 1: Alice and Betty each ante one dollar.

Step 2: Alice receives a random card face down and looks at it.

Step 3: Alice either bets one dollar or folds.

Step 4: Betty either calls (with one dollar) or folds.

Step 5: If both players are in the game, Alice wins the pot if she is holding a red card; otherwise Betty wins the pot.

**Questions:** If Alice gets a red card, should she ever fold? If Alice gets a black card, should she ever bluff (i.e., bet)?

**More difficult questions:** Can Alice make money from Betty, in the long term, by playing some strategy?

**Remarks on Fairness:** We can always make this game “fair” by having Alice and Betty play one round, and then having them play one round of the games with their roles reversed. Another way to make this game “fair” is to alter Alice’s ante and Betty’s ante.

1.2. **A Simple Matrix Game: Rock/Paper/Scissors.** We can represent rock/paper/scissors in a matrix giving Alice’s payout:

	B plays Rock	B plays Paper	B plays Scissors
A plays Rock	0	-1	1
A plays Paper	1	0	-1
A plays Scissors	-1	1	0

An  $m \times n$  *matrix game* can be viewed as a matrix,  $A = (a_{ij})$ , with  $m$  rows,  $n$  columns, where each row intuitively represents a “pure strategy” (or “move” or “play” etc.) by Alice, each column one by Betty, and the entry  $a_{ij}$  is how much money is awarded to Alice (and lost by Betty) when Alice plays pure strategy  $i$  and Betty plays pure strategy  $j$ .

1.3. **A Simpler Game: Even/Odd Pennies.**

Step 1: Alice and Betty each choose whether to place one or two pennies in their closed right hand.

Step 2: They open their hands. If the total number of pennies is even, Alice gets all the pennies; if odd, Betty gets them.

This yields

$$A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}.$$

The first row corresponds to Alice holding one penny, the second row to two pennies; similarly for columns and Betty.

In the literature there are many similar games named “even/odd pennies” or “matching pennies,” which differ from one another in the payout matrix or perhaps other rules.

1.4. **Some Examples.** Consider matrices like

$$\begin{bmatrix} 2 & 5 \\ 1 & 10 \end{bmatrix}, \quad \begin{bmatrix} 10 & 5 \\ 0 & 10 \end{bmatrix}, \quad \begin{bmatrix} 10 & -2 \\ -1 & 10 \end{bmatrix}.$$

Do you think Alice has an advantage in each game? What about

$$(1) \quad A = \begin{bmatrix} 10 & -25 \\ -1 & 10 \end{bmatrix} ?$$

A number of other matrix games are mentioned in [Was14], including “Holmes versus Moriarty” and two “investment problems.”

## 2. THE VALUES OF A MATRIX GAME

The amount of money that Alice and/or Betty will win in a matrix game depends on the rules. Consider the matrix game, “Simple,” with the following payouts to Alice:

	B plays Rock	B plays Paper
A plays Rock	1	-2
A plays Paper	-3	4

The *value* of this game depends on how it is played.

2.1. **Alice Announces a Pure Strategy.** Consider the following game which we will call “Alice Announces a Pure Strategy:”

- (1) first Alice chooses to play one of the two rows of the matrix;
- (2) then Alice announces this row to Betty;
- (3) then Betty picks a column; and
- (4) the corresponding matrix entry determines the payout to Alice.

Alice should pick the first row, so that Alice receives a payout of  $-2$  dollars (i.e., Betty gets 2 dollars from Alice); indeed, if Alice picks the second row, Betty will get 3 dollars from Alice.

Hence, in this game, the payout to Alice, given that Alice and Betty use their optimal strategy, is  $-2$  (to Alice).

2.2. **Betty Announces a Pure Strategy.** The game “Betty Announces a Pure Strategy” is the same game with roles reversed.

Betty does best to pick the first column, and Alice will then pick the first row. The value of this game is 1 (to Alice).

**2.3. Maxmin  $\leq$  Minmax.** The name of this subsection is the name of Subsection 2.2 in Washburn’s textbook [Was14]. The point is that the game “Betty Announces a Pure Strategy” has a value to Alice that is greater than or equal to “Alice Announces a Pure Strategy;” this will happen for any matrix game.

In the above example, “Betty Announces a Pure Strategy” has the value 1 to Alice, and “Alice Announces a Pure Strategy” has the value  $-2$ , which is less than the former value.

This also holds for “mixed strategies” (see below); it turns out that for mixed strategy games, the maxmin and minmax are always equal.

**2.4. Alice Announces a Mixed Strategy.** A mixed strategy for Alice means that Alice can play a fraction of row one and a fraction of row two (the fractions have to be non-negative and sum to one).

The game “Alice Announces a Mixed Strategy” means that Alice chooses a mixed strategy and announces her mixed strategy to Betty; then Betty chooses. For example, Alice can play 50% of row one and 50% of row two, and then Betty can choose either column. In matrix notation, Alice’s strategy is the row vector  $[\.5 \ .5]$ , which yields the row vector

$$[\.5 \ .5] \begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix} = [-1 \ 1] .$$

Then Betty can choose either column one or column two; at best, Betty chooses column one, and the value of this game is  $-1$  (the payout to Alice).

Alice’s above strategy can be described as Alice first tosses a fair coin, and then chooses the first row if the coin lands on “heads,” and the second row if the coin lands on “tails.” Betty is given this information, but is not allowed to see how the coin lands. One interpretation of the “value” is that the “expected” or “average” payout to Alice will be  $-1$  each time this game is played.

Notice that if Alice plays  $[\.7 \ .3]$  instead of  $[\.5 \ .5]$ , then Betty chooses between the two columns of

$$[\.7 \ .3] \begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix} = [-.2 \ -.2] .$$

Hence the value or payout to Alice of this game, with this mixed strategy, is  $-0.2$  to Alice. It turns out this mixed strategy is best for Alice. Hence the value of “Alice announces a mixed strategy” is  $-0.2$ .

**2.5. Betty Announces a Mixed Strategy.** One can define the analogous game “Betty Announces a Mixed Strategy.” If Betty uses a  $.5$ ,

.5 strategy, Alice will be choosing a row from:

$$\begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} .5 \\ .5 \end{bmatrix} = \begin{bmatrix} -.5 \\ .5 \end{bmatrix},$$

so Alice will choose the second row, giving her a payout of 0.5. If Betty announces a .6, .4 mixed strategy, Alice will be looking at

$$\begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} .6 \\ .4 \end{bmatrix} = \begin{bmatrix} -0.2 \\ -0.2 \end{bmatrix},$$

and Alice will get a payout of  $-0.2$  (with either choice of row for Alice). It turns out Betty can do no better than this. Hence the value of “Betty Announces a Mixed Strategy” is  $-0.2$ .

**2.6. The Duality Theorem for Matrix Games.** Matrix games are a special case of “linear programs,” which are the focus of Math 340. There is a “duality theory” that applies to all linear programs. In the case of matrix games it says that for any matrix, the games “Alice announces a mixed strategy” and “Betty announces a mixed strategy” have the same value. Let us formalize this theorem.

Given an  $m \times n$  matrix game,  $A$ , Alice has  $m$  pure strategies. A *mixed strategy* (or just *strategy*) for Alice is an  $m$ -dimensional stochastic vector, meaning a vector,  $\mathbf{x} = (x_i)$ , with components  $x_1, \dots, x_m$  with (1) each  $x_i \geq 0$ , and (2)  $\sum_i x_i = 1$ ; we can interpret  $x_i$  as the frequency with which Alice plays pure strategy  $i$  over many instances of the game, or her probability of playing pure strategy  $i$  in one game. The value of the game “Alice announces a mixed strategy” is

$$\text{AliceAnnouncesMixed}(A) = \max_{\mathbf{x} \text{ stoch}} \text{MinEntry}(\mathbf{x}^T A).$$

Similarly we have

$$\text{BettyAnnouncesMixed}(A) = \min_{\mathbf{y} \text{ stoch}} \text{MaxEntry}(A\mathbf{y}).$$

The duality theorem will imply that for any matrix  $A$  we have

$$\text{AliceAnnouncesMixed}(A) = \text{BettyAnnouncesMixed}(A).$$

### 2.7. The Definition of the (Mixed-Strategies) Value of a Matrix Game.

**Definition 2.1.** If  $A$  is an  $m \times n$  matrix of real numbers, then the (*mixed-strategies*) *value* of  $A$  (viewed as a matrix game) is the real number

$$\text{AliceAnnouncesMixed}(A) = \text{BettyAnnouncesMixed}(A).$$

We often drop the modifier *mixed-strategies*, and understand the *value* of a matrix game to be the mixed-strategies value.

Note that at present we don't know why the above two values are equal. After Chapter 5 of Vanderbei's text we will be able to prove this. (Of course, knowing a proof of something doesn't necessarily mean you know "why" it is true.)

**2.8. Summary of this Section.** To summarize this section, for each  $m \times n$  matrix,  $A$ , whose entries are real numbers, we can interpret  $A$  as a zero-sum game. We have looked at the four ways of playing the game:

- (1) Alice announces a pure strategy;
- (2) Betty announces a pure strategy;
- (3) Alice announces a mixed strategy; and
- (4) Betty announces a mixed strategy.

It should be intuitively clear that in terms of the value to Alice of each game we have

(2) Betty pure  $\geq$  (4) Betty mixed  $\geq$  (3) Alice mixed  $\geq$  (1) Alice pure .  
since

- (1) for Alice it is better for Betty to announce her strategy and then Alice gets to choose how to play, as opposed to having Alice announce, and
- (2) each player would rather be able to use a mixed strategy than forced to the more limited choice of a pure strategy.

In Chapter 5 of Vanderbei (duality theory for linear programs), we will see that

$$(4) \text{ Betty mixed} = (3) \text{ Alice mixed} ;$$

this is not obvious.

### 3. MORE ON POKER GAMES

In this section we will analyze the poker game of Section 1 and give a number of other poker games.

The poker game of Section 1 is described, in more general terms in pages 28–31 of [Was14]; our poker game corresponds to the game in [Was14] with  $p = 1/2$  and  $b = 1$ ; we therefore get the following payout to Alice, player 1, who plays against Betty, player 2 (see [Was14], Figure 3.8-2, page 30):

	B calls	B folds
A bets high, bets low	0	1
A bets high, folds low	1/2	0
A folds high, bets low	-3/2	0
A folds high, folds low	-1	-1

At the end of this section we will see how to derive this matrix, and do similarly for any poker game. Question: should Alice ever fold on high? We will answer this question in the following subsection.

### 3.1. Everywhere Better Rows and Columns.

**Definition 3.1.** Let  $A$  be a real,  $m \times n$  matrix. We say that one row of  $A$  is *everywhere better (for Alice)* than another row of  $A$  if each component of the first is at least as big as the corresponding component of the other.

In the above example, the row “A bets high, bets low” is everywhere better than “A folds high, bets low”; this means that we can delete the row “A folds high, bets low” from the poker game without hurting Alice, and hence without changing the value of the game.

We can make the analogous definition for columns of  $A$ , each corresponding to a pure strategy of Betty.

**Definition 3.2.** Let  $A$  be a real,  $m \times n$  matrix. We say that one column of  $A$  is *everywhere better (for Betty)* than another column of  $A$  if each component of the first is at least as *small* as the corresponding component of the other.

For example, in the matrix game

$$A = \begin{bmatrix} 10 & -25 \\ -1 & -12 \end{bmatrix}$$

the second column is everywhere better (for Betty) than the first column.

**3.2. The Payout Matrix in Poker Games.** Finding the payout matrix in a poker game is a bit tricky: it usually involves a number of payouts which depend on which cards each player gets. The matrix of payouts represent the *average* or *expected* payouts over the various possible dealings of the cards and their probabilities.

Let us consider the first poker game in this article (described in the beginning of Section 1. Here Alice is given a card, that is 50% of the time a high card, and 50% of the time a low card. Let us show how to find the payout matrix for various pure strategies of Alice and Betty.

**Example 3.3.** Consider the strategies “Alice bets on high, folds on low,” and “Betty calls.” This means that

- (1) 50% of the time Alice gets a high card; in this case Alice and Betty ante, Alice bets, Betty calls, and Alice wins 2 dollars from Betty (Betty’s ante and call);



- (2) 50% of the time Alice gets a low card; and in this case Alice and Betty ante, and Alice folds; hence in this case Betty wins 1 dollar from Alice (Alice's ante); in other words, the payout to Alice is  $-1$  dollar.

So the average payout to Alice is

$$(2) \quad (50\%)(2) + (50\%)(-1) = (0.5)(2) + (0.5)(-1) = 1/2.$$

This *average payout* is also called the *expected payout*.

**Example 3.4.** Let's analyze the expected payout to Alice for another pair of pure strategies for this poker game. Consider the strategies "Alice bets on high, bets on low," and "Betty calls." This means that

- (1) 50% of the time Alice gets a high card; in this case Alice and Betty ante, Alice bets, Betty calls, and Alice wins 2 dollars from Betty (Betty's ante and call);
- (2) 50% of the time Alice gets a low card; and in this case Alice and Betty ante, Alice bets, and then Betty calls; hence in this case the payout to Alice is  $-2$  dollars.

So the average payout to Alice is

$$(3) \quad (50\%)(2) + (50\%)(-2) = (0.5)(2) + (0.5)(-2) = 0.$$

This *average payout* is also called the *expected payout*.

**Example 3.5.** Let's change the poker game so that Alice wins only if she holds a heart (instead of a red card). In this case Alice gets a high card with probability  $1/4$  (i.e., 25% of the time), and a low card with probability  $3/4$  (i.e., 75% of the time). In this case, the first example above, where "Alice bets on high, folds on low," and "Betty calls," (2) gets modified to give

$$(25\%)(2) + (75\%)(-1) = (0.25)(2) + (0.75)(-1) = -1/4$$

as Alice's average payout. Similarly (3) becomes

$$(25\%)(2) + (75\%)(-2) = (0.25)(2) + (0.75)(-2) = -1.$$

**Example 3.6.** Consider a game where when Alice plays the pure strategy "Attempt 700 Pound Deadlift on Third Lift" and Betty plays the pure strategy "Attempt 725 Pound Deadlift on Third Lift" (or whatever), there is a 43% chance that Alice wins 500 dollars from Betty, a 28% chance that Alice wins 250 dollars from Betty, and a 29% chance that Alice loses 800 dollars to Betty. Then the average payout to Alice (or expected payout) is

$$(0.43)(500) + (0.28)(250) + (0.29)(-800) = 53.$$

In general, if any event has  $m$  outcomes, each of which has a certain payout to Alice, then the average or expected payout to Alice is

$$\begin{aligned} & (\text{Probability of outcome 1})(\text{Payout to Alice in case of outcome 1}) \\ & + (\text{Probability of outcome 2})(\text{Payout to Alice in case of outcome 2}) \\ & + \cdots + (\text{Probability of outcome } m)(\text{Payout to Alice in case of outcome } m). \end{aligned}$$

#### 4. REMARKABLE THEOREMS

Before discussing linear programming, we will describe a few remarkable theorems about matrix games. After going through the basics of linear programming and duality theory (Chapters 1–5 of Vanderbei’s text), it will be easy to prove all these theorems.

**Theorem 4.1** (First Remarkable Theorem). *Let  $A$  be any real valued  $m \times n$  matrix. Then*

$$\text{AliceAnnouncesMixed}(A) = \text{BettyAnnouncesMixed}(A).$$

This was stated earlier. This is not easy to prove; once we know duality theory (Chapter 5 of Vanderbei), this will be easy to show.

There are a number remarkable theorems about the optimal strategies for the games “Alice Announces a Mixed Strategy” and “Betty Announces a Mixed Strategy.”

**Definition 4.2.** Let  $A$  be a real valued  $m \times n$  matrix and  $\mathbf{x} \in \mathbb{R}^m$  a stochastic vector. (In other words,  $\mathbf{x}$  is a vector of  $m$  non-negative reals that sums to one; and  $\mathbf{x}$  can be interpreted as a mixed strategy for Alice who plays a stochastic or “mixed strategy” of rows of  $A$ .)

We define

$$\text{AliceAnnouncesMixed}(A; \mathbf{x}) = \text{MinEntry}(\mathbf{x}^T A).$$

We say that  $\mathbf{x}^*$  is an *optimal mixed strategy for Alice* if

$$\text{AliceAnnouncesMixed}(A; \mathbf{x}^*) = \max_{\mathbf{x} \text{ stoch}} \text{AliceAnnouncesMixed}(A; \mathbf{x});$$

in other words,  $\mathbf{x}^*$  is a “best” mixed strategy for Alice.

We similarly define, for a stochastic  $\mathbf{y} \in \mathbb{R}^n$

$$\text{BettyAnnouncesMixed}(A; \mathbf{y}) = \text{MaxEntry}(A\mathbf{y}),$$

and we say that  $\mathbf{y}^*$  is an *optimal mixed strategy for Betty* if  $\mathbf{y}^*$  is a vector for which  $\text{BettyAnnouncesMixed}(A; \mathbf{y})$  is maximized over all stochastic,  $n$ -dimensional vectors  $\mathbf{y}$ .

In practice we are interested in finding a best mixed strategy for Alice and Betty.

**Definition 4.3.** We say that  $A \in \mathbb{R}^{m \times n}$  (i.e., a real,  $m$  by  $n$  matrix) is *irreducible* if every optimal mixed strategy for Alice is positive on all its components, and the same with “Alice” replaced with “Betty.”

For example, the matrix

$$A = \begin{bmatrix} 3 & 5 \\ 3 & 6 \end{bmatrix}$$

is not irreducible, since an optimal mixed strategy for Alice is to play only the top row; another optimal mixed strategy is for Alice to play only the bottom row (in both cases Betty will choose the first column). The matrix

$$A = \begin{bmatrix} 3 & 5 \\ 4 & 4 \end{bmatrix}$$

is also not irreducible, since Alice’s best strategy is to play only the bottom row.

**Theorem 4.4** (An Easy Theorem). *If  $A$  is not irreducible, then one can delete some row or some column for  $A$  and get a new matrix whose value is the same.*

**Theorem 4.5** (Second Remarkable Theorem). *If  $A$  is an irreducible matrix, then (1)  $A$  must be a square matrix, and (2) Alice has a unique optimal strategy, and (3) Betty has a unique optimal strategy.*

## 5. TWO BY TWO MATRIX GAMES

A retired UBC math prof, Klaus Hochsmann, used to teach linear algebra by spending the first two weeks of the course doing all of the course in the special case of  $2 \times 2$  matrix. Here we borrow his idea, and describe how to compute the value of any  $2 \times 2$  matrix game. The  $2 \times 2$  case illustrates a number (but not all) of the fundamental principles of matrix games.

Two by two matrix games fall into two classes.

**Theorem 5.1.** *Let  $A \in \mathbb{R}^{2 \times 2}$ . Then either:*

- (1)  *$A$  is not irreducible; in this case the value of “Alice announces a pure strategy” and “Betty announces a pure strategy” are the same; or*
- (2)  *$A$  is irreducible, and the unique optimal strategy for Alice is the unique solution,  $\mathbf{x}$ , to*

$$\mathbf{x}^T A = [v \ v]$$

for a real number  $v$  and  $\mathbf{x}$  such that  $x_1 + x_2 = 1$  (we will automatically see that  $x_1, x_2 > 0$  in this case); similarly the unique optimal strategy for Betty is the solution to

$$A\mathbf{y} = \begin{bmatrix} v \\ v \end{bmatrix}.$$

**Example 5.2.** Let

$$A = \begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix}$$

The value of “Alice announces a pure strategy” is  $-2$ , and the value of “Betty announces a pure strategy” is  $1$ . Since these two values are not equal, we have that  $A$  is irreducible. Hence we should have a unique solution  $\mathbf{x}$  and  $v$  to the equations

$$\mathbf{x}^T A = [v \ v] \quad x_1 + x_2 = 1,$$

or

$$x_1 - 3x_2 = v, \quad -2x_1 + 4x_2 = v, \quad x_1 + x_2 = 1;$$

the first two equations tell us that

$$x_1 - 3x_2 = -2x_1 + 4x_2, \quad \text{i.e.,} \quad 3x_1 = 7x_2;$$

so  $3x_1 = 7x_2$  and  $x_1 + x_2 = 1$  gives

$$x_1 = 0.7, \quad x_2 = 0.3,$$

and  $v = x_1 - 3x_2 = -0.2$ . Alternatively we may solve for Betty’s optimal strategy:

$$A\mathbf{y} = \begin{bmatrix} v \\ v \end{bmatrix}, \quad y_1 + y_2 = 1;$$

this gives

$$y_1 - 2y_2 = v, \quad -3y_1 + 4y_2 = v, \quad y_1 + y_2 = 1$$

which gives  $y_1 = .6$ ,  $y_2 = .4$ ,  $v = -0.2$ .

Of course, we know that the  $v$  we find for Alice’s optimal mixed strategy has to be the same as that for Betty’s optimal mixed strategy.

**Example 5.3.** Consider

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

The value of “Alice announces a pure strategy” is  $3$  (Alice will pick the second row, which is everywhere better for Alice than the first row), and the value of “Betty announces a pure strategy” is  $3$  (Betty will pick the first column, which is everywhere better for Betty than the second column). These two values are  $3$ , and  $A$  is not irreducible.

## 6. TERMINOLOGY

In this section we give some terminology that is (1) in common use outside of this course, and/or (2) had a different name in earlier versions of this course. The point in (2) is that previous exam questions were asked with slightly different terminology.

Here is some such terminology. Let  $A \in \mathbb{R}^{m \times n}$  be viewed as a matrix game.

- We say that one row of  $A$  *dominates* another row of  $A$  if the first row is everywhere better (for Alice) than the second row.
- Similarly, we say that one column of  $A$  *dominates* another if the first column is everywhere better than the second column.
- If  $\mathbf{x}^*$  is an optimum mixed strategy for Alice, and  $\mathbf{y}^*$  is an optimum mixed strategy for Betty, then we call the pair  $(\mathbf{x}^*, \mathbf{y}^*)$  an *equilibrium* of  $A$ .
- We define the *duality gap* of  $A$  to be the value of “Betty announces a pure strategy” minus the value of “Alice announces a pure strategy;” this number is always non-negative.
- In earlier versions of these notes we used the term “Alice screams” for “Alice announces a mixed strategy” and “Alice announces” for “Alice announces a pure strategy.”

Actually, the notion of an *equilibrium* is a general notion valid in many more two-person games. We may explain this a bit more in class.

Also the notion of a *duality gap* is also very general. We may explain this in more detail in class, probably at the end of the class.

## 7. THE DUALITY GAP

In this section we define the very general notion of a “duality gap.” We will probably only cover this at the very end of the course.

**Definition 7.1.** Let  $X, Y$  be sets, and  $f: X \times Y \rightarrow \mathbb{R}$  an arbitrary function. We define

$$\text{BettyAnnounces}(f) = \min_{y \in Y} \left( \max_{x \in X} f(x, y) \right)$$

(this is commonly called the *min-max* of  $f$ )

$$\text{AliceAnnounces}(f) = \max_{x \in X} \left( \min_{y \in Y} f(x, y) \right),$$

(this is commonly called the *max-min* of  $f$ ), and the *duality gap* of  $f$  to be

$$\text{DualityGap}(f) = \text{BettyAnnounces}(f) - \text{AliceAnnounces}(f).$$

A few remarks are in order:

- (1) The duality gap is always non-negative; this is intuitively clear if you think of the game “Betty Announces a  $y \in Y$ ,” where Betty chooses a value of  $y \in Y$  and announces it to Alice, who then chooses a value  $x \in X$  and gets a payout of  $f(x, y)$ , as opposed to the similarly defined “Alice Announces an  $x \in X$ .” Clearly “Betty Announces” is at least as good to Alice as “Alice Announces.”
- (2) To make things rigorous, one has to replace “max” by “supremum” and “min” by the “infimum.” In the two cases of interest we have either (1)  $X$  and  $Y$  are finite sets, or (2)  $X$  and  $Y$  are compact topological spaces and  $f$  is continuous; in both of these cases it is correct to write “max” and “min.”

**Theorem 7.2.** *Let  $X, Y$  be arbitrary sets, and let  $f$  be an arbitrary real-valued function on  $X \times Y$ . Then the duality gap of  $f$  is non-negative.*

**Example 7.3.** Let  $X$  be the pure states of Alice, and  $Y$  the pure states of Betty in an  $m \times n$  matrix game with underlying matrix  $A$ . Then for each  $x \in X$  and  $y \in X$  we set  $f(x, y) = A_{xy}$ . Then the duality gap of  $f$  is our notion of the duality gap of  $A$ .

**Example 7.4.** Let  $A \in \mathbb{R}^{m \times n}$ ; let  $X$  be the set of stochastic vectors in  $\mathbb{R}^m$ , and let  $Y$  be the set of stochastic vectors in  $\mathbb{R}^n$ . For  $\mathbf{x} \in X$  and  $\mathbf{y} \in Y$  define

$$f(\mathbf{x}, \mathbf{y}) = \mathbf{x}^T A \mathbf{y}.$$

Then our duality theorem regarding matrix games implies that the duality gap of  $f$  is zero.

## 8. THE BIG POKER QUESTION: A $2^{52} \times 2$ POKER GAME

At the end of this course, we will try to analyze the following poker game. First we order the cards in the poker order:

$$A\spadesuit > A\heartsuit > A\diamondsuit > A\clubsuit > K\spadesuit > K\heartsuit > \dots > 2\diamondsuit > 2\clubsuit$$

So for any two cards in a standard 52 card deck, one card is deemed “higher” than the other.

- Step 1: Alice and Betty each ante one dollar (we might change the ante of both players).
- Step 2: Alice receives a random card face down and looks at it.
- Step 2.5: Betty receives a random card face down from the deck (now having 51 cards left), but neither Alice nor Betty can look at this second card.
- Step 3: Alice either bets one dollar or folds.

Step 4: Betty either calls (with one dollar) or folds.

Step 5: If both players are in the game, Alice wins the pot if she is holding a higher card than Betty; otherwise Betty wins the pot.

Notice that Betty still has only two strategies: “fold” or “call,” since Betty cannot see either of the cards. However Alice can choose “fold” or “bet” on each of 52 different cards, giving her  $2^{52}$  different pure strategies. This gives us a  $2^{52} \times 2$  matrix game. We know that from all the  $2^{52}$  strategies of Alice, we can essentially ignore all but two of Alice’s strategies, according to our remarkable theorems. The question is: which are the two strategies?

Will we be able to answer this by the end of term?

## 9. LEARNING GOALS AND SAMPLE EXAM PROBLEMS

It seems best to make general learning goals concrete by connecting them with sample exam problems.

### Learning Goals:

- (1) You should know how to calculate the *expected* (i.e., the *average*) payout to someone in games of chance such as card games. See Exercise 9.1.
- (2) You should know what the entries of a matrix mean when viewed as a matrix game; you should be able to write down the matrix when given the payouts of a game. See Exercise 9.3, 9.4, 9.5, 9.6, 9.10.
- (3) You should know that *symmetric* matrix games satisfy  $A = -A^T$  and be able to explain why this is true. You should know their value. See Exercise 9.5, 9.6, 9.10.
- (4) You should know how to (quickly) find the value of the games “Alice announces a pure strategy” and “Betty announces a pure strategy” for a matrix (viewed as a matrix game). See Exercise 9.2.
- (5) You should know that the value of “Alice announces a mixed strategy” in any matrix game is the same as “Betty announces a mixed strategy,” and that this will be proven using “Duality Theory” (Chapter 5 of Vanderbei’s text).
- (6) You should be able to compute the value of a matrix game and the optimal mixed strategies of Alice and of Betty; you should know the method of Section 5 to quickly do this for a

$2 \times 2$  matrix, first finding the duality gap and, if it is positive, solving a system of equations. See Exercise 9.7, 9.8, 9.11.

### Sample Exam Problems

**Exercise 9.1.** (The following questions deal with calculating the expected or average payout to Alice without any strategies for Alice or Betty.) Alice gets dealt a card from a standard, 52-card deck, at random. Compute the expected (average) payout to Alice if:

- (1) Betty pays Alice one dollar if Alice draws a red card, and Betty pays Alice four dollars if Alice draws a black card.
- (2) If Alice draws a spade, she must pay Betty four dollars; otherwise Betty pays Alice three dollars.
- (3) If Alice draws an ace, she must pay Betty four dollars; otherwise Betty pays Alice three dollars.
- (4) Alice pays Betty one dollar if she draws a 7 or an 8 of any suit, and otherwise no one pays anything to the other player.
- (5) Betty pays Alice the number of dollars that Alice's card shows if it shows the number 2, 3, 4,  $\dots$ , 10, but Alice pays Betty five dollars if Alice's card is an Ace, King, Queen, or Jack.

**Exercise 9.2.** Use Section 2 to find the value of the two games: (1) Alice announces a pure strategy, and (2) Betty announces a pure strategy, for the matrices:

(1)

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$$

(2)

$$A = \begin{bmatrix} -1 & -3 \\ -2 & -5 \end{bmatrix}$$

(3)

$$A = \begin{bmatrix} 1 & 3 \\ 5 & 2 \end{bmatrix}$$

(4)

$$A = \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}$$

(5)

$$A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \\ 5 & 3 \end{bmatrix}$$



(6)

$$A = \begin{bmatrix} 4 & 1 & 2 \\ 7 & 2 & 3 \\ 5 & 3 & 6 \end{bmatrix}$$

**Exercise 9.3.** Consider a matrix,  $A$ , that is viewed as a matrix game. Let  $A'$  be  $A$  with its first two rows exchanged.

- (1) Is the value of  $A'$  the same as that of  $A$  in the matrix game “Alice announces a pure strategy?” Explain.
- (2) Is the value of  $A'$  the same as that of  $A$  in the matrix game “Alice announces a mixed strategy?” Explain.
- (3) Is Alice’s optimum strategy (or strategies) in the game  $A'$  the same as that of  $A$  in the matrix game “Alice announces a mixed strategy?” Explain.
- (4) Is Betty’s optimum strategy (or strategies) in the game  $A'$  the same as that of  $A$  in the matrix game “Betty announces a mixed strategy?” Explain.
- (5) What does the matrix game  $A'$  represent in terms of  $A$  in terms of the pure strategies of Alice?
- (6) What does the matrix game  $A'$  represent in terms of  $A$  in terms of the pure strategies of Betty?

**Exercise 9.4.** Let  $A$  be a matrix, and let  $2A$  denote the matrix obtained by multiplying each entry of  $A$  by 2.

- (1) Is the mixed-strategy value of  $2A$  always twice that of  $A$ ? Explain.
- (2) How do the other values and optimal strategies of  $2A$  relate to those of  $A$ ? Explain.
- (3) Can you say the same about  $-2A$ ? Explain, giving examples to show why or why not.

**Exercise 9.5.** Consider an  $m \times n$  matrix,  $A$ , viewed as the payout to Alice in a matrix game between Alice playing the rows and Betty playing the columns. Explain why if Alice and Betty exchange roles, the new payout matrix (which is the payout to Betty with Betty playing the rows) is the  $n \times m$  matrix  $-A^T$ .

**Exercise 9.6.** Consider a square matrix,  $A$ , such that  $A = -A^T$ . Explain why  $A$ , when viewed as a matrix, is called a *symmetric* matrix game. Give an example of a symmetric matrix game.

**Exercise 9.7.** Compute the duality gap, the mixed-games values, and optimal mixed strategies for Alice and for Betty for the  $2 \times 2$  matrix games of Exercise 9.2, (1)–(4).

**Exercise 9.8.** The first poker game in this article yields the payout to Alice given in the beginning of Section 3:

	B calls	B folds
A bets high, bets low	0	1
A bets high, folds low	$1/2$	0

(We already know that Alice should never fold on a high card, so Alice really has only two strategies; we also found that the value of this game is  $1/3$ .) Consider the variant of this game: before the game starts, Alice pays Betty a fee of  $\alpha$  dollars to play the game, where  $\alpha$  is some positive real number. Then the game proceeds as before. What value of  $\alpha$  makes the game fair?

**Exercise 9.9.** Consider the variant of our first poker game (see above) where Alice’s ante is  $\alpha$  instead of 1; everything else is the same.

- (1) Show that the payout to Alice when she plays “bets high, bets low” and Betty plays “fold” is still 1.
- (2) Show that the payout to Alice when she plays “bets high, bets low” and Betty plays “call” is  $1/2 - \alpha/2$  [Hint: there is 50% a chance that Alice gets a high card, and 50% that she gets a low card; if Alice draws a high card she wins 2 from Betty, but if Alice gets a low card, she loses  $1 + \alpha$  to Betty.]
- (3) Show that the rest of the payout to Alice matrix looks like:

	B calls	B folds
A bets high, bets low	$1/2 - \alpha/2$	1
A bets high, folds low	$1 - \alpha/2$	$1/2 - \alpha/2$

- (4) Compute the value of this game for all values of  $\alpha \geq 1$ . For which  $\alpha$  is the above game fair?

**Exercise 9.10.** Consider a *symmetric* matrix game, i.e.,  $-A^T = A$ , i.e., Alice and Betty have the same set of “pure strategies,” and the payout to Alice is the same if Alice and Betty change roles.

- (1) Explain why Rock-Paper-Scissors is a symmetric game, and verify that the payout-to-Alice matrix satisfies  $-A^T = A$ .
- (2) Argue that an optimal strategy for “Alice announces a pure strategy” is the same as that for Betty.
- (3) Argue that an optimal strategy for “Alice announces a mixed strategy” is the same as that for Betty.
- (4) Show that the value of the game “Alice announces a mixed strategy” is 0.

**Exercise 9.11.** Consider a variant of rock/paper/scissors in which a rock beating scissors pays 4 times as much as the other two situations.

- (1) What would you guess is the optimal mixed strategy for Alice?  
[There is no right or wrong answer here; hopefully your guess is not the correct answer, or you will learn less than you would otherwise.]
- (2) Calculate the optimal mixed strategy for Alice.
- (3) Give a short explanation for why this mixed strategy is optimal (that does not refer to your calculation).

**Exercise 9.12.** Consider the following variant on the first poker game of Section 1:

- Steps 1–4 are the same.
  - Step 5, which determines the payout is as follows:
  - If Alice has a heart, then she wins the pot. If Alice has diamond, then she wins the pot and Betty has to give Alice an extra dollar. If Alice has a club or a spade, then Betty wins the pot.
- (1) Argue that Betty has two strategies, “fold” and “call.”
  - (2) Argue that Alice has eight strategies, such as “fold on heart, bet on diamond, fold on black card” and “bet on heart, fold on diamond, fold on black card.”
  - (3) Argue that Alice should never fold on any red card, and that therefore we need only consider two strategies.
  - (4) Write the expected payout matrix to Alice, based on Alice’s two relevant strategies, and Betty’s two strategies.
  - (5) Compute the value of this poker game and the optimal mixed strategies of Alice and Betty.

**Exercise 9.13.** Consider the following variant on the first poker game of Section 1:

- Steps 1–4 are the same.
  - Step 5, which determines the payout to Alice is as follows:
  - If Alice has a red card, then she wins the pot. If Alice has club, then Betty wins the pot. If Alice has a spade, then Betty wins the pot and Alice has to give Betty one additional dollar.
- (1) Argue that Betty has two strategies, “fold” and “call.”
  - (2) Argue that Alice has eight strategies, such as “fold on red card, bet on spade, fold on club” and “bet on red card, fold on spade, bet on club.”
  - (3) Argue that Alice should never fold on any red card, and that therefore we need only consider four strategies.

- (4) Write the expected payout matrix to Alice, based on Alice's four relevant strategies, and Betty's two strategies.
- (5) Compute the value of this poker game and the optimal mixed strategies of both players.

**Exercise 9.14.** Explain why the value (to Alice) of “Alice announces a pure strategy” is no more than that of “Alice announces a mixed strategy.”

**Exercise 9.15.** Explain why the value (to Alice) of “Alice announces a mixed strategy” is no more than that of “Betty announces a mixed strategy.”

**Exercise 9.16.** Let

$$A = \begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix}, \quad A' = \begin{bmatrix} 2 & -1 \\ -2 & 5 \end{bmatrix}.$$

Given that

$$A' = A + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

interpret the game  $A'$  in terms of  $A$ . How do the values of the four games involving  $A$  compare to the four games involving  $A'$ ? How do the optimum strategies for Alice and Betty compare in the various games?

## APPENDIX A. EXERCISES BEYOND EXAM QUESTIONS

In this section I will put exercises that are more difficult than what you would get on an exam.

If you solve any of these and would like me to see your solutions, please write them up separately from the homework and hand them directly to me (rather than put them in a pile for homework).

**Exercise A.1.** Let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

be a  $2 \times 2$  matrix viewed as a matrix game, where all of its entries are integers. Let

$$D = a_{11} + a_{22} - a_{12} - a_{21}.$$

- (1) Show that if  $D = 0$ , then  $A$  is reducible (i.e., not irreducible).
- (2) Show that if  $D \neq 0$  and if  $A$  is reducible, then the optimal mixed strategies for Alice and for Betty, and the value of the game, are all given by integers divided by  $D$ . [Hint: you could obtain formulas for the optimal strategies and the value of the game; but you should be able to give a quicker argument that

all of these equations give answers that are integers divided by  $D$ .]

## APPENDIX B. REVIEW OF MATRIX NOTATION

In this appendix we review matrix notation; this makes mixed strategies more concise. Later in this course the matrix notation will be a crucial tool to understanding and improving the simplex method.

Consider the first poker game we studied:

$$A = \begin{bmatrix} 0 & 1 \\ 1/2 & 0 \\ -3/2 & 0 \\ -1 & -1 \end{bmatrix}$$

It is customary to write the entries of  $A$  as

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{bmatrix}$$

so that  $a_{ij}$  represents the entry of  $A$  in the  $i$ -th row and  $j$ -th column. If Alice plays the first row with probability  $.6$  and the last row with probability  $.4$  (and hence Alice never plays the second and third rows), then Alice is playing

$$(.6)[0 \ 1] + (.4)[-1 \ -1] = [-.4 \ .2],$$

which can also be written in matrix multiplication as:

$$[.6 \ 0 \ 0 \ .4] \begin{bmatrix} 0 & 1 \\ 1/2 & 0 \\ -3/2 & 0 \\ -1 & -1 \end{bmatrix}$$

If Alice announces this to Betty, this is what Betty is choosing from. Similarly, if Betty calls  $.3$  of the time and folds  $.7$  of the time, then Betty is playing

$$\begin{bmatrix} 0 & 1 \\ 1/2 & 0 \\ -3/2 & 0 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} .3 \\ .7 \end{bmatrix} = \begin{bmatrix} .7 \\ .15 \\ -.45 \\ -1 \end{bmatrix}$$

We can write the four games associated to  $A$  in matrix notation:

- (1) Alice announces a pure strategy: here if Alice declares that she will play for  $i$ , Betty chooses a column in

$$[a_{i1} \ a_{i2} \ \dots \ a_{in}],$$

and Betty will evidently choose

$$\min_j a_{ij}.$$

Alice wants to choose the row to maximize this, the value of this game to Alice is

$$\max_i \left( \min_j a_{ij} \right).$$

- (2) Betty announces a pure strategy: similarly the value to Alice of this game will be

$$\min_j \left( \max_i a_{ij} \right).$$

- (3) Alice announces a mixed strategy: now Alice chooses a stochastic vector  $\mathbf{x}$  as her mixed strategy (so  $\mathbf{x}$  has non-negative components that sum to one), and Betty will pick the smallest value from among the components of

$$\mathbf{x}^T A.$$

So the value to Alice of this game is

$$\max_{\mathbf{x} \text{ stoch.}} \text{MinEntry}(\mathbf{x}^T A).$$

- (4) Betty announces a mixed strategy: similarly the value to Alice of this strategy is:

$$\min_{\mathbf{y} \text{ stoch.}} \text{MaxEntry}(A\mathbf{y}).$$

Now let's try to compute the value of "Alice announces a mixed strategy." Note that Alice will never play the third or fourth row in an optimal strategy, so we are essentially playing the game

$$\begin{bmatrix} 0 & 1 \\ 1/2 & 0 \end{bmatrix}$$

Note that

$$\begin{aligned}
 [1 \ 0] \begin{bmatrix} 0 & 1 \\ 1/2 & 0 \end{bmatrix} &= [0 \ 1] & [.9 \ .1] \begin{bmatrix} 0 & 1 \\ 1/2 & 0 \end{bmatrix} &= [.05 \ .9] \\
 [.8 \ .2] \begin{bmatrix} 0 & 1 \\ 1/2 & 0 \end{bmatrix} &= [.10 \ .8] & [.7 \ .3] \begin{bmatrix} 0 & 1 \\ 1/2 & 0 \end{bmatrix} &= [.15 \ .7] \\
 [.6 \ .4] \begin{bmatrix} 0 & 1 \\ 1/2 & 0 \end{bmatrix} &= [.20 \ .6] & [.5 \ .5] \begin{bmatrix} 0 & 1 \\ 1/2 & 0 \end{bmatrix} &= [.25 \ .5] \\
 [.4 \ .6] \begin{bmatrix} 0 & 1 \\ 1/2 & 0 \end{bmatrix} &= [.30 \ .4] & [.3 \ .7] \begin{bmatrix} 0 & 1 \\ 1/2 & 0 \end{bmatrix} &= [.35 \ .3] \\
 [.2 \ .8] \begin{bmatrix} 0 & 1 \\ 1/2 & 0 \end{bmatrix} &= [.40 \ .2] & [.1 \ .9] \begin{bmatrix} 0 & 1 \\ 1/2 & 0 \end{bmatrix} &= [.45 \ .1] \\
 [.0 \ 1.0] \begin{bmatrix} 0 & 1 \\ 1/2 & 0 \end{bmatrix} &= [.50 \ .0]
 \end{aligned}$$

We note that the minimum component of the above vectors  $\mathbf{x}^T A$  is no more than 0.3, which is achieved for

$$[.4 \ .6] \begin{bmatrix} 0 & 1 \\ 1/2 & 0 \end{bmatrix} = [.30 \ .4] \quad [.3 \ .7] \begin{bmatrix} 0 & 1 \\ 1/2 & 0 \end{bmatrix} = [.35 \ .3]$$

So it seems like the optimal  $\mathbf{x}$  is somewhere between  $[.4 \ .6]$  and  $[.3 \ .7]$ . By linear algebra we can perfectly balance things, solving for

$$[p_1 \ p_2] \begin{bmatrix} 0 & 1 \\ 1/2 & 0 \end{bmatrix} = [v \ v], \quad p_1 + p_2 = 1,$$

which is solved uniquely by  $p_1 = 1/3$  and  $p_2 = 2/3$  (and  $v = 1/3$ ).

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