HOMEWORK 3 SOLUTIONS

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Problem 1

(a) (There are a few ways of doing this): Since Alice takes the stochastic $\mathbf{x}$ whose smallest component is as large as possible, any particular $\mathbf{x}$ is at least as good (in terms of its smallest component) as the best $\mathbf{x}$. Said otherwise: the value of "Alice announces a mixed strategy is

$$\text{AliceAnnouncesMixed}(A) = \max_{\mathbf{x} \text{ stoch}} \text{MinEntry}(\mathbf{x}^{\mathrm{T}} A);$$

hence this is at least as large as any particular value of

$$\text{MinEntry}(\mathbf{x}^{\mathrm{T}} A)$$

for a particular stochastic vector, $\mathbf{x}$.

(b) Similarly any particular choice of stochastic $\mathbf{y}$ gives a MaxEntry($A\mathbf{y}$) larger than

$$\text{BettyAnnouncesMixed}(A) = \min_{\mathbf{y} \text{ stoch}} \text{MaxEntry}(A\mathbf{y}).$$

(c) (Again, there are a number of possible solutions): In the game "Betty announces a mixed strategy," Alice can always ignore what Betty announces and play a mixed strategy $\mathbf{x}$, and then let Betty revise her strategy if she wants; this cannot be worse for Betty. But this game is just "Alice announces a mixed strategy."

Alternatively, one can see that

$$v_1 \le (\mathbf{x}^{\mathrm{T}} A)\mathbf{y},$$

since $\mathbf{y}$ is stochastic and each entry of $\mathbf{x}^{\mathrm{T}} A$ is at least v_1. Similarly

$$v_2 \ge \mathbf{x}^{\mathrm{T}}(A\mathbf{y}).$$

Hence

$$v_1 \le \mathbf{x}^{\mathrm{T}} A\mathbf{y} \le v_2.$$

(d) The value of "Alice announces a mixed strategy" cannot be greater than v_1 if $v_1 = v_2$, by part (a). Since Alice can achieve a value of v_1, this is the maximum of "Alice announces a mixed strategy.

(e) We have

$$[2/3\ 1/3] \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} = [0\ 0],$$

Research supported in part by an NSERC grant.

and so $v_1 \geq 0$; we have

$$\begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

and so $v_1 \leq 0$. By part (d), it follows that 0 is the value of this game.

(f) If $\mathbf{x}^{\mathrm{T}}A$ has its minimum component, v, equal to the maximum component, w, of $A\mathbf{y}$, then by part (d) this common number $v = w$ must be the value of the mixed strategy games (and $\mathbf{x}$ and $\mathbf{y}$ are optimum strategies). If not, then either $\mathbf{x}$ or $\mathbf{y}$ is not an optimal strategy, since we know that the two mixed strategy games have the same value.

Problem 2

We have the same initial dictionary:

$$\begin{array}{ccccc} \zeta & = & & 4x_1 & +5x_2 \\ \hline w_1 & = & 8 & -x_1 & -2x_2 \\ w_2 & = & 5 & -x_1 & -x_2 \\ w_3 & = & 8 & -2x_1 & -x_2 \end{array}$$

If x_1 enters, then the most restrictive inequality is $w_3 \geq 0$ (since this imposes that $x_1 \leq 4$, rather than the first two which impose, respectively, $x_1 \leq 5$ and $x_1 \leq 8$); hence x_1 enters forces w_3 to leave, and we get the second dictionary:

$$\begin{array}{ccccc} \zeta & = & 16 & -2w_3 & +3x_2 \\ \hline w_1 & = & 1 & +(1/2)w_3 & -(1/2)x_2 \\ w_2 & = & 4 & +(1/2)w_3 & -(1/2)x_2 \\ x_1 & = & 4 & -(1/2)w_3 & -(1/2)x_2 \end{array}$$

Now x_2 must enter (since w_3 has a negative coefficient in the ζ line); w_1 leaves since it is the most restrictive, and we get the third dictionary:

$$\begin{array}{ccccc} \zeta & = & 22 & +w_3 & -6w_1 \\ \hline x_2 & = & 2 & +w_3 & -2w_1 \\ w_2 & = & 1 & -w_3 & +3w_1 \\ x_1 & = & 3 & -w_3 & +w_1 \end{array}$$

So w_3 enters, and w_2 leaves, to yield the fourth dictionary

$$\begin{array}{ccccc} \zeta & = & 23 & -w_2 & -3w_1 \\ \hline x_2 & = & 3 & -w_2 & +w_1 \\ w_3 & = & 1 & -w_2 & +3w_1 \\ x_1 & = & 2 & +w_2 & -2w_1 \end{array}$$

Since the coefficients of the non-basic variables in the ζ row are all negative or zero, we are done. We get $(x_1, x_2) = (2, 3)$ gives the largest possible value of $\zeta = 4x_1 + 5x_2$, namely 23.

Problem 3

(a) Our first dictionary is

$$\begin{array}{rcllll} \zeta & = & & v & & \\ \hline w_1 & = & & -v & +11x_1 & +8x_2 \\ w_2 & = & & -v & +9x_1 & +12x_2 \\ w_3 & = & 1 & & -x_1 & -x_2 \end{array}$$

v enters, and either w_1 or w_2 leaves; let's choose w_1 (this is a degenerate pivot, where the ζ value does not increase and the basic feasible solution remains the same).

$$\begin{array}{rcllll} \zeta & = & & -w_1 & 11x_1 & +8x_2 \\ \hline v & = & & -w_1 & +11x_1 & +8x_2 \\ w_2 & = & & w_1 & -2x_1 & +4x_2 \\ w_3 & = & 1 & & -x_1 & -x_2 \end{array}$$

Now we can take x_1 entering or x_2. Say we choose x_1 enters, so w_2 leaves (in another degenerate pivot).

$$\begin{array}{rcllll} \zeta & = & 0 & +(9/2)w_1 & -(11/2)w_2 & +30x_2 \\ \hline v & = & 0 & +(9/2)w_1 & -(11/2)w_2 & +30x_2 \\ x_1 & = & 0 & +(1/2)w_1 & -(1/2)w_2 & +2x_2 \\ w_3 & = & 1 & -(1/2)w_1 & +(1/2)w_2 & -3x_2 \end{array}$$

Now let's choose x_2 to enter (we could choose w_1 as well, which would ultimately add one more pivot) to obtain:

$$\begin{array}{rcllll} \zeta & = & 10 & -10w_3 & -(1/2)w_2 & -(1/2)w_1 \\ \hline v & = & 10 & -10w_3 & -(1/2)w_2 & -(1/2)w_1 \\ x_1 & = & 2/3 & -(2/3)w_3 & -(1/6)w_2 & +(1/6)w_1 \\ x_2 & = & 1/3 & -(1/3)w_3 & +(1/6)w_2 & -(1/6)w_1 \end{array}$$

This shows that Alice's optimal mixed strategy is $(x_1, x_2) = (2/3, 1/3)$, giving a value of 10.

(b) Our starting dictionary is:

$$\begin{array}{rcllll} \zeta & = & & v & & \\ \hline w_1 & = & & -v & -9x_1 & -12x_2 \\ w_2 & = & & -v & -11x_1 & -8x_2 \\ w_3 & = & 1 & & -x_1 & -x_2 \end{array}$$

So v enters and w_1 or w_2 leaves; say we take w_1: we get degenerate pivot to the final dictionary:

$$\begin{array}{rcllll} \zeta & = & 0 & -1w_1 & -9x_1 & -12x_2 \\ \hline v & = & 0 & -1w_1 & -9x_1 & -12x_2 \\ w_2 & = & 0 & +1w_1 & -2x_1 & +4x_2 \\ w_3 & = & 1 & +0w_1 & -1x_1 & -1x_2 \end{array}$$

(c) Since all entries of A_{+10} are positive, it is clear that $v > 0$, so $v \geq 0$ is no real restriction. Furthermore if we allow $x_1 + x_2 \leq 1$, it will not change the optimum value of v, since any solution with $x_1 + x_2 < 1$ can have either x_1 or x_2 increased to get a larger value of v

(d) We know that Alice's optimal mixed strategy is given by the same LP in part (a) except with $x_1 + x_2 = 1$ and v arbitrary; by part (c) we know that these changes don't change the optimum mixed strategy.

(e) A_{+10} is the game where Betty gives Alice \$10 and then they play the game A. Since this initial payment is independent of the rest of the game, the equilibria are the same and the value of A_{+10} is exactly 10 more than that of A. Similarly for A_{-10}.

(f) The discussion in part (c) does not apply, for v is certainly negative, and setting $x_1 + x_2 \le 1$ allows $x_1 = x_2 = 0$ (intuitively allowing Alice not to play at all), which is better for v and Alice in a game with all negative entries. (There are other possible explanations for this question.)

(g) Betty's strategy is in the ζ row of the final dictionary. We will understand this later (Chapter 5) as due to the fact that the dictionary dual to Alice's final dictionary is a final dictionary for Betty.

For this small dictionary one may be able argue differently, without appealing to duality theory (Chapter 5). For example, if we want a linear combination of the columns that "balances" (Betty's equilibrium), this balancing can be done by finding the combination of w_1 and w_2 whose x_1, x_2 coefficients are equal, therefore recovering v with $10w_3$. In other words, the slack variables w_1 and w_2 each correspond to a column, and balancing them in terms of the slack w_3 corresponds to column balancing.

Perhaps you have another (and even simpler) explanation?

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