HOMEWORK 2 SOLUTIONS

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Exercise 9.4

(1) and (2): (There are many ways to doing this problem.) Multiplying by 2 is equivalent to dividing the currency by 2. Hence all the strategies are the same, and each game value for the new matrix game is twice that of the corresponding original value.

(3) There is no simple relationship between the matrix games A and $-A$; for an example, see items (1) and (2) of the next exercise.

Exercise 9.7(1)

Alice announces a pure strategy: since the minimum entry of row 1 is 1, and the minimum entry of row 2 is 2, Alice chooses row 2, which gives a value of 2 to this game.

Betty announces a pure strategy: since the maximum entry of column 1 is 2, and the maximum entry of row 2 is 5, Betty chooses column 1, which gives a value of 2 to this game.

Hence the duality gap is $2-2=0$. Since the pure strategy games have the same value, the values of the mixed games are also 2, and optimal mixed strategies are just the pure strategies above.

Exercise 9.7(2)

Alice announces a pure strategy: similarly to the above reasoning (see also the previous problem set), Alice chooses row 1, with a value of -3.

Betty announces a pure strategy: Betty chooses column 2, with a value of -3.

Hence the duality gap is $-3-(-3)=0$. Again, since the pure strategy games have the same value, the values of the mixed games are also -3, and optimal mixed strategies are just the pure strategies above.

Exercise 9.7(3)

Reasoning as above, the value of Alice announces a pure strategy is 2, and of Betty announces a pure strategy is 3. Hence the duality gap is $3-2=1$, and

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since this is positive we know that A is irreducible; and we see that Alice's optimal strategy $[p_1\ p_2]$ is given by

$$[p_1\ p_2]\begin{bmatrix} 1 & 3 \\ 5 & 2 \end{bmatrix} = [v\ v], \quad p_1+p_2=1,$$

so

$$p_1+5p_2=v=3p_1+2p_2$$

so $3p_2=2p_1$, so we get $p_1=3/5$ and $p_2=2/5$ and $v=13/5$. Similarly Betty's optimal strategy is $[q_1\ q_2]$ where

$$\begin{bmatrix} 1 & 3 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} v \\ v \end{bmatrix}, \quad q_1+q_2=1,$$

and we find that $q_1=1/5$, $q_2=4/5$, $v=13/5$.

Exercise 9.7(4)

Alice announces a pure strategy has value 0, since she chooses the first row, and Betty announces a pure strategy has value 0, since she chooses the first colomn. Then the duality gap is zero, and the above pure strategies are also optimal mixed strategies.

Exercise 9.8

In class we have seen that the value of this game is 1/3. Therefore, to get a fair game, i.e., a game of value 0, we have Alice pay a fee of 1/3 to Betty.

Exercise 9.12

(1) As with similar poker games we have seen, Betty must play either "fold" or "call" when Alice is still in the game (and otherwise Betty has nothing to do). Since Betty has no information other than the fact that Alice is still in the game, Betty has only those strategies.

(2) Alice has the options of betting or folding on each of three possible card types: hearts, diamonds, or a black suit, for a total of $2^3=8$ possible strategies.

(3) If Alice folds on any red card, she will gain more money by staying in the game, no matter what Betty does. Hence the two strategies where Alice bets on any red card dominate the other six strategies.

(4) The only difference from the originial poker game is that if Alice bets on a red card and Betty calls, then 1/4 of the time she gets an extra dollar. Hence we add 1/4 to each entry where Betty calls, yielding the payout matrix:

$$A=\begin{bmatrix} 1/4 & 1 \\ 3/4 & 0 \end{bmatrix}$$

For this matrix the duality gap is 3/4 (namely the value of Betty announces, Betty picking the first column) minus 1/4 (namely the value of Alice announcing the first row), which is positive, so we know that A is irreducible and we solve for Alice's optimal mixed strategy via:

$$[p_1\ p_2]\begin{bmatrix} 1/4 & 1 \\ 3/4 & 0 \end{bmatrix} = [v\ v], \quad p_1+p_2=1,$$

so

$$p_1 = p_2 = 1/2, \quad v = 1/2.$$

Similarly Betty's optimal strategy is $[q_1\ q_2]$ where

$$\begin{bmatrix} 1/4 & 1 \\ 3/4 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} v \\ v \end{bmatrix}, \quad q_1 + q_2 = 1,$$

and we find that $q_1 = 2/3$, $q_2 = 1/3$, $v = 1/2$.

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