Solutions: Homework #7

1. Problem 3.5:

   (a) By adding the same constant, C, to each c_{ijk} you simply add $3nC$ to the total utility, without affecting which matching is best.

   (b) This is essentially the same argument done in class for 2-dimensional matching: assume that you are working with the inequalities where each $=$ is replaced by a $\leq$. We claim that in optimality one cannot have
$$\sum_{jk} x_{ijk} < 1$$
for some $i = i_0$ (the case where the summation is over i, k for fixed $j = j_0$, or i, j for fixed $k = k_0$, is argued similarly): if so, then we have
$$\sum_{ijk} x_{ijk} < n,$$
and hence there exists a j_0 for which
$$\sum_{ik} x_{ij_0k} < 1$$
and a k_0 for which
$$\sum_{ij} x_{ijk_0} < 1.$$
In this case we may add some small amount to $x_{i_0j_0k_0}$ without violating the inequalities; since the c_{ijk} are all strictly positive, this increase would increase the utility, which violates the assumed optimality of the solution.

   (c) In this linear program we get a feasible solution by taking all the x_{ijk} to be zero. The linear program is bounded by n times the maximum value among the c_{ijk}.

(d) Since this linear program has $3n$ constraints, all dictionaries have $3n$ basic variables, and hence at most $3n$ nonzero variables in any BFS (Basic Feasible Solution) associated to any dictionary of the simplex method. Hence in the optimal dictionary, which gives the optimal solution, we have at most $3n$ nonzero values.

2. Problem 3.6:

(a) The columns associated to x_1 and x_2 in the "big A" matrix are the same; since A_B is invertible, we cannot have both x_1 and x_2 basic.

(b) The initial dictionary is

$$\begin{aligned} \zeta &= 4x_1 + 5x_2 \\ w_1 &= 10 - x_1 - x_2 \\ w_2 &= 21 - 2x_1 - 2x_2 \\ w_3 &= 29 - 3x_1 - 3x_2 \end{aligned}$$

Say that x_1 enters; then w_3 leaves and we get

$$\begin{aligned} \zeta &= 116/3 - (4/3)w_3 + x_2 \\ w_1 &= 1/3 - (1/3)w_3 \\ w_2 &= 4/3 - (2/3)w_3 \\ x_1 &= 29/3 - (1/3)w_3 - x_2 \end{aligned}$$

Then x_2 enters and x_1 leaves, which gives

$$\begin{aligned} \zeta &= 145/3 - (5/3)w_3 - x_1 \\ w_1 &= 1/3 - (1/3)w_3 \\ w_2 &= 4/3 - (2/3)w_3 \\ x_2 &= 29/3 - (1/3)w_3 - x_1 \end{aligned}$$

which is an optimal dictionary. If x_2 enters the dictionary first then we immediately get the same optimal dictionary.

The reasoning in subquestion (1) does not depend on the constants 10, 21, and 29; hence this conclusion is independent of these constants.

(Note that in all dictionaries, if x_1 or x_2 is in the basis, then the other is nonbasic and appears only the the x_i dictionary row, always with a -1 coefficient; we explained why this is true in class when we had variables that could be positive or negative, so each such variable had to be written as the difference of two non-negative variables; in this case, if one difference is $t_1 - t_2$, then then t_1 column in "big A" is minus that of t_2.)

3. Problem 3.7:

   (a) Since the data points don't lie on a single line, we must have $d > 0$. By adding $C + Cx$ to each data point for C large, the new optimal will have the new a and b equal to C plus the old a and b. [Note that at this point we don't know how large we need C to be.]

   Since a, b, d can be assumed to be non-negative, we get a linear program in standard form as:

$$
\begin{array}{ll}
\text{maximize } \zeta = -d, \quad \text{subject to} & \\
\qquad\qquad -a - d & \leq \; -4, \\
\qquad\qquad a - d & \leq \; 4, \\
\qquad\qquad -b - a - d & \leq \; -6, \\
\qquad\qquad b + a - d & \leq \; 6, \\
\qquad\qquad -2b - a - d & \leq \; -7, \\
\qquad\qquad 2b + a - d & \leq \; 7, \\
\qquad\qquad -3b - a - d & \leq \; -10, \\
\qquad\qquad 3b + a - d & \leq \; 10, \\
\text{and} \qquad\qquad b, a, d \geq 0. &
\end{array}
$$

   (b) We may take any value of a, b, and then take d to be the maximum absolute value of $|y - a - bx|$ taken over all data points (x, y); these values of a, b, d are feasible. Clearly d is bounded below by zero.

   (c) It follows that in any optimal dictionary a, b, d must be basic, so at least three of the slack variables must be nonbasic and hence zero in the optimal solution. (Since we cannot have both $y - a - bx$ equalling d and $-d$ for a data point (x, y) (since $d \neq 0$ any any solution), the three slack variables are associated to three different data points.)

(d) The above arguments do not depend on the number of data points, as long as the data points do not all lie on one line (so that d cannot be 0). (This implies that there are at least three data points, or two data points with the same x value but different y values.)

4. Problem 3.9: We have w_1, w_2, w_3, w_4 are slack variables with $w_1+w_2 = w_3 + w_4$. For any dictionary, the basic variables, B, must be uniquely expressible in terms of the nonbasic variables (since A_B is invertible). But since $w_1 + w_2 - w_3 - w_4 = 0$, if w_1, w_2, w_3, w_4 were all nonbasic, then one could add any multiple of $w_1 + w_2 - w_3 - w_4$ to any entry in the dictionary, showing that one can express any basic variable in infinitely many ways in the dictionary. Hence w_1, w_2, w_3, w_4 cannot all be nonbasic.

   Here is another proof: since $w_1+w_2 = w_3+w_4$, it follows that if we take row 1 of "big A" and add to it row 2 and subtract rows 3 and 4, we get a matrix that is row equivalent to "big A" but is zero in row 1 outside of the coefficients in the w_1, w_2, w_3, w_4 columns. So if w_1, w_2, w_3, w_4 were all non-basic, A_B would be row equivalent to a matrix with all zeros in its first row; this contradicts the fact that A_B is invertible.