Solutions: Homework #6

1. Problem 3.1:

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 0 & 0 & 5 \\ 1 & 2 & 0 & 1 & 0 & 8 \\ 2 & 1 & 0 & 0 & 1 & 8 \\ \hline 4 & 5 & 0 & 0 & 0 & 0 \end{array}\right],$$

x_1 enters, so $w_3 = x_5$ is the most restrictive and it leaves; so we divide row 3 by 2

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 0 & 0 & 5 \\ 1 & 2 & 0 & 1 & 0 & 8 \\ 1 & 1/2 & 0 & 0 & 1/2 & 4 \\ \hline 4 & 5 & 0 & 0 & 0 & 0 \end{array}\right],$$

and then we clear out all column 1 entries on rows not equal to 3, i.e., we subtract row 3 from row 1

$$\left[\begin{array}{ccccc|c} 0 & 1/2 & 1 & 0 & -1/2 & 1 \\ 1 & 2 & 0 & 1 & 0 & 8 \\ 1 & 1/2 & 0 & 0 & 1/2 & 4 \\ \hline 4 & 5 & 0 & 0 & 0 & 0 \end{array}\right],$$

and similarly subtract row 3 from row 2 and subtract 4 times row 3 from row 4 (the ζ row)

$$\left[\begin{array}{ccccc|c} 0 & 1/2 & 1 & 0 & -1/2 & 1 \\ 0 & 3/2 & 0 & 1 & -1/2 & 4 \\ 1 & 1/2 & 0 & 0 & 1/2 & 4 \\ \hline 0 & 3 & 0 & 0 & -2 & -16 \end{array}\right].$$

This finishes one pivot; now, looking at the ζ row, we see that x_2 must enter; the most restrictive constraint from all rows is row 1, and this corresponds to the basic variable $x_3 = w_1$, so $x_3 = w_1$ leaves. So we divide row 1 by 1/2,

$$\left[\begin{array}{ccccc|c} 0 & 1 & 2 & 0 & -1 & 2 \\ 0 & 3/2 & 0 & 1 & -1/2 & 4 \\ 1 & 1/2 & 0 & 0 & 1/2 & 4 \\ \hline 0 & 3 & 0 & 0 & -2 & -16 \end{array}\right]$$

and then clear out everything in column 2 in rows $2, 3, 4$:

$$\left[\begin{array}{ccccc|c} 0 & 1 & 2 & 0 & -1 & 2 \\ 0 & 0 & -3 & 1 & 1 & 1 \\ 1 & 0 & -1 & 0 & 1 & 3 \\ \hline 0 & 0 & -6 & 0 & 1 & -22 \end{array}\right].$$

Now the ζ row shows that x_5 must enter the basis, and the most restrictive is row 2, corresponding to $x_4 = w_2$ which leaves; so we use the 1 in column 5 of row 2 to clear out anything in column 5 either above or below row 2:

$$\left[\begin{array}{ccccc|c} 0 & 1 & -1 & 1 & 0 & 3 \\ 0 & 0 & -3 & 1 & 1 & 1 \\ 1 & 0 & 2 & -1 & 0 & 2 \\ \hline 0 & 0 & -3 & -1 & 0 & -23 \end{array}\right].$$

This gives the same final tableau as before.

2. Problem 3.2: The first tableau is

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 0 & 0 & 3 \\ 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1 & 2 \\ \hline 1 & 7 & 0 & 0 & 0 & 0 \end{array}\right].$$

Taking x_2 to enter, the most restrictive is the third row, so we pivot on the third row, column 2:

$$\left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & -1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1 & 2 \\ \hline 1 & 0 & 0 & 0 & -7 & -14 \end{array}\right];$$

the associated eta matrix to affect these row operations in the non-zeta rows (i.e., rows 1–3) is

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now x_1 enters, and the most restrictive is row 1, so we pivot on row 1, column 1:

$$\left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & -1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 2 \\ \hline 0 & 0 & -1 & 0 & -6 & -15 \end{array}\right];$$

the associated eta matrix to affect these row operations is

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now x_1 enters, and the most restrictive is row 1, so we pivot on This is a final dictionary, giving $x_2 = 2$ and $x_1 = 1$ as the optimal solution.

3. Problem 3.4.

(1) Given that we obtain each $A_{B_{i+1}}^{-1}$ from an eta matrix multiplied by $A_{B_i}^{-1}$, if we have already computed $A_{B_i}^{-1}$ (or made some computation to allow ourselves to multiply vectors on the left and right by $A_{B_i}^{-1}$), then the additional cost of multiplying by an eta matrix of dimension 100×100 would be an addition 200 FLOPS (floating point operations). So for the first few steps we would definitely want to use the repeated multiplication by a few eta matrices to determine the parts of the dictionary that we need to look at.

(2) By the same principle, the cost of repeatedly multiplying by 500 eta matrices would be roughly 500 times 200 FLOPS; on the other hand, multiplying a 100×100 matrix (of the inverse of any 100×100 matrix) by a column or row vector would take roughly 100 times 200 FLOPS (once the inverse is recomputed, i.e., once the matrix is refactored). Of course would take roughly 100^3 time to factor a general 100×100 matrix, but the claim in the Vanderbei is that often this is much smaller for the types of matrices one uses in practice (see Section 8.4).

In Section 8.4 it is claimed explained that one recomputes $A_{B_i}^{-1}$ for about every 100 or so interations, although a theoretical calculation of the number of FLOPS used shows that if the number of FLOPS needed to refactorize $A_{B_i}^{-1}$ is m^3, you should refactorize every m iterations, and if it is m^2 (which happens in some sparser problems), you should refactorize every $\sqrt{m}$ iterations (where the eta matrices are $m \times m$). The effect of roundoff error is not discussed in Section 8.4.

In any event, it is clear that once one computes $A_{B_i}^{-1}$ (probably implicitly by computing an LU-decomposition), for the next few iterations it is advantageous to use eta matrices, and for a large number of interations eta matrices will be problematic (either in roundoff error or, after a very long time, because it is cheaper to recompute $A_{B_i}^{-1}$). The details of exactly how often one should refactor will depend on the particular problem.