Solutions: Homework #5

1. (a) The value is between 1 and 16 since these are the smallest and largest entries of A.

   (b)

$$\begin{array}{ll} \text{maximize } \zeta = v, & \\ & -v + p_1 + 4p_2 + 9p_3 + 16p_4 \le 0, \\ \text{subject to} & -v + 16p_1 + 9p_2 + 4p_3 + p_4 \le 0, \\ & p_1 + p_2 + p_3 + p_4 \le 1 \\ \text{and} & v, p_1, p_2, p_3, p_4 \ge 0. \end{array}$$

   Here we write $p_1 + \cdots + p_4 \le 1$ because any feasible solution to the above where $p_1 + \cdots + p_4 < 1$ remains feasible when we increase any of $p_1, \ldots, p_4$ and hold v fixed (since all the entries of A are positive). Hence it suffices to write $p_1 + \cdots + p_4 \le 1$.

   (c) If we begin the simplex method on the above linear program, then we will have three slack variables, and hence each dictionary will have three basic variables. Since the constraints all have non-negative constant terms, the initial dictionary is feasible and hence the linear program is feasible; since the objective, $\zeta = v$, is at most 16, this linear program is bounded; hence the simplex method will take us to a final dictionary. Since v (the value of the game) is between 1 and 16, v must be a basic in the final dictionary. Hence there are two other basic variables in the final dictionary, and all other variables take on the value zero in BFS (basic feasible solution) associated to the final dictionary. Hence there is an optimal solution to the above linear program, which gives Alice's optimal mixed strategy, where at most two of $p_1, \ldots, p_4$ are non-zero.

   (d) We see that the value of Alice announces a pure strategy is 1, and of Betty announces a pure strategy is 16. Hence there is a positive duality gap, and we find Alice's optimal mixed strategy (since this is a 2×2 matrix game) by solving

$$[q_1 \ q_2]A' = [v \ v]$$

and $q_1+q_2=1$; we get the equations

$$q_1+16q_2=v=16q_1+q_2, \quad q_1+q_2=1$$

whose unique solution is $q_1=q_2=1/2$ and $v=17/2$.

Hence Alice's optimal mixed strategy for A' is $[1/2\ 1/2]$. We similarly find Betty's optimal mixed strategy for A' to be $[1/2\ 1/2]$.

(e) We see that if Alice plays $[1/2\ 0\ 0\ 1/2]$ in the original 4×2 game, we have

$$[1/2\ 0\ 0\ 1/2]A=[17/2\ 17/2],$$

so for this particular strategy, Betty can do no better than 17/2. On the other hand, if Betty plays $[1/21/2]$ (her strategy for A'), then Alice will choose among

$$A\begin{bmatrix}1/2\\1/2\end{bmatrix}=\begin{bmatrix}17/2\\13/2\\13/2\\17/2\end{bmatrix},$$

so Alice will get a payout of 17/2.

Since we have found a mixed strategies for Alice and Betty that give the same payout when Alice announces her mixed strategy to Betty as when Betty announces hers to Alice, these strategies must be optimal, with this common payout being the value of the (mixed strategy) game.

(f) First solve the 2×2 matrix game where we discard all of Alice's rows except for rows number 12 and 29. Then take Betty's optimal (mixed) strategy for the 2×2 game, multiply it by the 100×2 matrix, and see if any row has a value greater than the value in both rows 12 and 29: if no, then your intuition (that Alice has an optimal strategy involving only rows 12 and 29) is correct; otherwise it is incorrect.

2. The initial dictionary is

$$\begin{array}{rcrr}
\zeta & = & & x_1 \\ \hline
w_1 & = & 5 & -x_1 \\
w_2 & = & -2 & +x_1
\end{array}$$

Since w_2 has a negative constant, we have to use the two-phase method. According to the instructions, we have to add x_0 to each equation:

$$\begin{array}{lcrrr}
\zeta_{\text{new}} & = & & & -x_0 \\
\hline
w_1 & = & 5 & -x_1 & +x_0 \\
w_2 & = & -2 & +x_1 & +x_0
\end{array}$$

Since w_2 requires x_0 to be at least 2, and there are no other constraints, x_0 enters and w_2 leaves, yielding

$$\begin{array}{lcrrr}
\zeta_{\text{new}} & = & -2 & +x_1 & -w_2 \\
\hline
w_1 & = & 7 & -2x_1 & +w_2 \\
x_0 & = & 2 & -x_1 & +w_2
\end{array}$$

Looking at ζ_{new}, x_1 must enter, and $x_0 \geq 0$ imposes $x_1 \leq 2$, which is more restrictive than the condition $w_1 \geq 0$ (which imposes $x_1 \leq 7/2$); hence as x_1 enters the basis, x_0 leaves:

$$\begin{array}{lcrrr}
\zeta_{\text{new}} & = & & -x_0 & \\
\hline
w_1 & = & 3 & +2x_0 & -w_2 \\
x_1 & = & 2 & -x_0 & +w_2
\end{array}$$

So we get a feasible dictionary for the original linear program (and original objective, $\zeta = x_1$) by removing the x_0:

$$\begin{array}{lcrr}
\zeta & = & 2 & +w_2 \\
\hline
w_1 & = & 3 & -w_2 \\
x_1 & = & 2 & +w_2
\end{array}$$

This is the start of phase two; we now pivot using the usual simplex method, using the original objective $\zeta = x_1$. So w_2 enters the basis, and w_1 leaves:

$$\begin{array}{lcrr}
\zeta & = & 5 & -w_1 \\
\hline
w_2 & = & 3 & -w_1 \\
x_1 & = & 5 & -w_1
\end{array}$$

Hence we get to the final dictionary, finding the optimal feasible solution $x_1 = 5$ where $\zeta = 5$ (and $w_1 = 0$ and $w_2 = 3$).