Solutions: Homework #4

1.

$$\begin{array}{rcrrr} w_1 & = & 1 & -x_1 & -x_2 \\ w_2 & = & -1 & -2x_1 & +x_2 \\ w_3 & = & -4 & +3x_1 & +2x_2 \\ \hline \zeta & = & & 3x_1 & +x_2 \end{array}$$

This is not feasible; we use the two-phase method. Introduce x_0 and maximize $w = -x_0$:

$$\begin{array}{rcrrrr} w_1 & = & 1 & -x_1 & -x_2 & +x_0 \\ w_2 & = & -1 & -2x_1 & +x_2 & +x_0 \\ w_3 & = & -4 & +3x_1 & +2x_2 & +x_0 \\ \hline \zeta & = & & & & -x_0 \end{array}$$

The most negative variable is w_3, so x_0 enters and w_3 leaves.

$$\begin{array}{rcrrrr} x_0 & = & 4 & -3x_1 & -2x_2 & +w_3 \\ w_1 & = & 5 & -4x_1 & -3x_2 & +w_3 \\ w_2 & = & 3 & -5x_1 & -x_2 & +w_3 \\ \hline \zeta & = & -4 & +3x_1 & +2x_2 & -w_3 \end{array}$$

x_1 enters and w_2 leaves:

$$\begin{array}{rcrrrr} x_0 & = & 11/5 & -(7/5)x_2 & +(3/5)w_2 & +(2/5)w_3 \\ x_1 & = & 3/5 & -(1/5)x_2 & -(1/5)w_2 & +(1/5)w_3 \\ w_1 & = & 13/5 & -(11/5)x_2 & +(4/5)w_2 & +(1/5)w_3 \\ \hline \zeta & = & -11/5 & +(7/5)x_2 & -(3/5)w_2 & -(2/5)w_3 \end{array}$$

x_2 enters and w_1 leaves

$$\begin{array}{rcrrrr} x_0 & = & 6/11 & +(7/11)w_1 & +(1/11)w_2 & +(3/11)w_3 \\ x_1 & = & 4/11 & +(1/11)w_1 & -(3/11)w_2 & +(2/11)w_3 \\ x_2 & = & 13/11 & -(5/11)w_1 & +(4/11)w_2 & +(1/11)w_3 \\ \hline \zeta & = & -6/11 & -(7/11)w_1 & -(1/11)w_2 & -(3/11)w_3 \end{array}$$

The maximum value of $\zeta = -6/11$. Hence the minimum value of $x_0 = 6/11$. Since this is non-zero, we conclude that the original LP problem is not feasible.

2. Again, we introduce slacks and then x_0 to the right-hand-sides.

$$\begin{array}{rcrrr} w_1 & = & 7 & -x_1 & +x_0 \\ w_2 & = & -1 & +x_1 & +x_0 \\ w_3 & = & -4 & +x_1 & +x_0 \\ \hline \zeta & = & & & -x_0 \end{array}$$

(You could omit the x_0 from the w_1 line; the dictionaries would be slightly different.) We pivot x_0 in and w_3, which has the most negative constant, leaves.

$$\begin{array}{rcrrr} w_1 & = & 11 & -2x_1 & +w_3 \\ w_2 & = & 3 & & +w_3 \\ x_0 & = & 4 & -x_1 & +w_3 \\ \hline \zeta & = & -4 & +x_1 & -w_3 \end{array}$$

So x_1 enters and x_0 leaves.

$$\begin{array}{rcrrr} w_1 & = & 3 & +2x_0 & -w_3 \\ w_2 & = & 3 & & +w_3 \\ x_1 & = & 4 & -x_0 & +w_3 \\ \hline \zeta & = & & -x_0 & \end{array}$$

So x_0 is eliminated and we bring in the old objective:

$$\begin{array}{rcrr} w_1 & = & 3 & -w_3 \\ w_2 & = & 3 & +w_3 \\ x_1 & = & 4 & +w_3 \\ \hline \zeta & = & 12 & +3w_3 \end{array}$$

So w_3 enters and w_1 leaves and we get:

$$\begin{array}{rcrr} w_3 & = & 3 & -w_1 \\ w_2 & = & 6 & -w_1 \\ x_1 & = & 7 & -w_1 \\ \hline \zeta & = & 21 & -3w_1 \end{array}$$

This is a final dictionary, so we are done.

Had we pivoted w_2 instead of x_0 into the first dictionary, we would have obtained

$$\begin{array}{rcrrr} w_1 & = & 8 & -2x_1 & +w_2 \\ x_0 & = & 1 & -x_1 & +w_2 \\ w_3 & = & -3 & & +w_2 \\ \hline \zeta & = & -1 & +x_1 & -w_2 \end{array}$$

In this case we get a negative constant in the w_3 row.

3. (a)

$$\begin{array}{rcrrr} w_1 & = & 2+\epsilon_1 & -x_1 & \\ w_2 & = & 3+\epsilon_2 & & -x_2 \\ w_3 & = & 5+\epsilon_3 & -x_1 & -x_2 \\ \hline \zeta & = & & 2x_1 & +x_2 \end{array}$$

x_2 enters and w_2 leaves.

$$\begin{array}{rcrrr} w_1 & = & 2+\epsilon_1 & -x_1 & \\ x_2 & = & 3+\epsilon_2 & & -w_2 \\ w_3 & = & 2-\epsilon_2+\epsilon_3 & -x_1 & +w_2 \\ \hline \zeta & = & 3+\epsilon_2 & +2x_1 & -w_2 \end{array}$$

x_1 enters; since $2+\epsilon_1 > 2-\epsilon_2+\epsilon_3$ (we'd have a tie if the ϵ_i's weren't there), w_3 leaves:

$$\begin{array}{rcrrr} w_1 & = & \epsilon_1+\epsilon_2-\epsilon_3 & +w_3 & -w_2 \\ x_2 & = & 3+\epsilon_2 & & -w_2 \\ x_1 & = & 2-\epsilon_2+\epsilon_3 & -w_3 & +w_2 \\ \hline \zeta & = & 7-\epsilon_2+2\epsilon_3 & -2w_3 & +w_2 \end{array}$$

Now w_2 enters and w_1 leaves (this would be a degenerate pivot if not for the ϵ_i's).

$$\begin{array}{rcrrr} w_2 & = & \epsilon_1+\epsilon_2-\epsilon_3 & +w_3 & -w_1 \\ x_2 & = & 3-\epsilon_1+\epsilon_3 & -w_3 & +w_1 \\ x_1 & = & 2+\epsilon_1 & & -w_1 \\ \hline \zeta & = & 7+\epsilon_1+\epsilon_3 & -w_3 & -w_1 \end{array}$$

This dictionary is final, and so the optimal solution is $x_1 = 2$, $x_2 = 3$. A picture is given Figure 1.

(b) The dictionaries look the same, with ϵ_1 and ϵ_3 interchanged. So the first dictionary is

$$\begin{array}{rcrrr} w_1 & = & 2+\epsilon_3 & -x_1 & \\ w_2 & = & 3+\epsilon_2 & & -x_2 \\ w_3 & = & 5+\epsilon_1 & -x_1 & -x_2 \\ \hline \zeta & = & & 2x_1 & +x_2 \end{array}$$

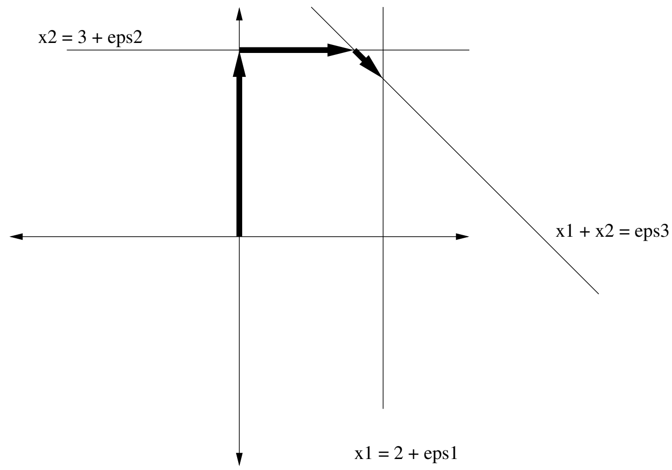

Figure 1: Drawing for the first simplex method

and the first pivot is x_2 enters, w_2 leaves, as before. But on the second pivot x_1 enters and w_1 leaves, since the the second dictionary looks like:

$$\begin{array}{rcrrr} w_1 & = & 2+\epsilon_3 & -x_1 & \\ x_2 & = & 3+\epsilon_2 & & -w_2 \\ w_3 & = & 2-\epsilon_2+\epsilon_1 & -x_1 & +w_2 \\ \hline \zeta & = & 3+\epsilon_2 & +2x_1 & -w_2 \end{array}$$

and $2+\epsilon_3 < 2-\epsilon_2+\epsilon_1$. Thus we get to the final dictionary

$$\begin{array}{rcrrr} w_3 & = & -\epsilon_3-\epsilon_2+\epsilon_1 & +w_1 & +w_2 \\ x_2 & = & 3+\epsilon_2 & & -w_2 \\ x_1 & = & 2+\epsilon_3 & -w_1 & \\ \hline \zeta & = & 7+2\epsilon_3+\epsilon_2 & -2w_1 & -w_2 \end{array}$$

A picture is give in Figure 2.

(c) Note that this final dictionary is different, with different basic variables! Also, we took one fewer pivots this new way.

Note also that this shows there is more than one final dictionary here. For the future, this means that there is more than one optimal solution for the dual problem; this is due to the three lines coinciding at the optimal solution for the primal LP.

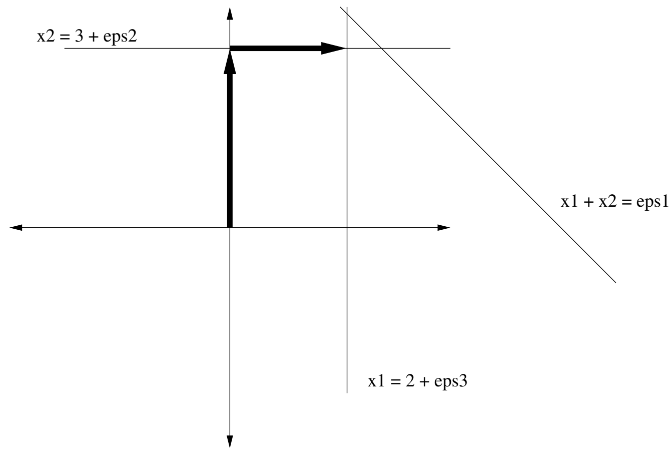

Figure 2: Drawing for the second simplex method, i.e., ϵ_1 and ϵ_3 interchanged

When we perturb a degenerate problem, we can expect more than one geometry to emerge depending on the perturbation. The perturbed problems and the ϵ_i's are guides to which pivots to take, so different geometries give rise to different sequences of dictionaries.

There are many other comments one could make along these lines (this is not a threat).