Math 223, Starting Jan 21

- Section 2 of Articles ties into textbook (starting Jan 21)

Jan 21

Goal: Let $S = T = \text{Functions}(\mathbb{Z} \to \mathbb{R})$, let $\mathcal{L}: S \to T$ be given as

$$(\mathcal{L}f)(n) = f(n+2) - f(n+1) - f(n).$$

Goal: Show that $\ker(\mathcal{L})$ is a 2-dimensional vector space.

Why?

$$\text{Fibonacci}(n) \in \ker(\mathcal{L})$$

but also

$$\ldots, r^{-2}, r^{-1}, 1, r, r^2, \ldots \in \ker(\mathcal{L})$$

for

$$r = r_+ \text{ or } r_-, \quad \text{where } r_+ = \frac{1+\sqrt{5}}{2}, \quad r_- = \frac{1-\sqrt{5}}{2}.$$

Want to explain that

$$\text{Fibonacci} = \frac{1}{\sqrt{5}}\left((r_+)^n - (r_-)^n\right)$$

And, more generally:

$\ker(\mathcal{L})$ is spanned by $(r_+)^n$ and $(r_-)^n$.

Need some terminology from textbook by Jänich...

Here's what we'll use from the textbook over the next while:

From Jänich:

§2.1 (Section 2.1) Real Vector Spaces:

- page 17: Usual definition of Vector Space

(examples: $\mathbb{R}^n$, $M :=$ Functions$([-1,1] \to \mathbb{R})$)

(Jänich writes $(\mathbb{R}^n, +, \cdot)$, $(M, +, \cdot)$, etc.)

§2.2 Complex etc. — We only use this for the operators

$(\mathcal{L}f)(n) = f(n+4) - f(n)$ homework and similar operators

§2.3 Subspaces

Skip the rest of Ch. 2 for now

Ch. 3 Dimension

§3.1. Linear independence

page 43: Span = Linear hull = Linear combinations

$$L(v_1, \ldots, v_r) = \{ \lambda_1 v_1 + \ldots + \lambda_r v_r \mid \lambda_i \in \mathbb{F} \} \subset V$$

we write as

$$\text{Span}(v_1, \ldots, v_r) \subset V$$

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page 45: $v_1, \ldots, v_n \in V$ is a basis if (1) Lin indep (p.44)
(2) Span $= V$ (p.43)

page 45: Canonical basis for $\mathbb{R}^n$: $\vec{e}_1, \dots, \vec{e}_n$

§3.2 If a vector space, $\bar{V}$, over $\mathbb{R}$ has a basis $v_1, \dots, v_n$ (Jänich writes $(v_1, \dots, v_n)$), then any other basis has n elements. <u>Dimension</u> of $\bar{V}$ is n.

(Theorem 1, p. 46)

Proof by "basis exchange lemma"

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3.5: The "vector product" means $\times$ product in $\mathbb{R}^3$ is standard but disappointing for an <u>honours</u> course.

Jan 21 (still...)

Recall: $S = T =$ Function $(\mathbb{Z} \to \mathbb{R})$ or $\text{Poly}_{\leq 3}(\mathbb{R})$ or...

To say: given $\mathcal{L} : S \to T$

$$\mathcal{L}(s_1) = \mathcal{L}(s_2) \text{ (in } T) \iff \mathcal{L}(s_1 - s_2) = 0_T$$

needed $-$ in S $(-_S)$, $-$ in T $(-_T)$ and

(1) $\mathcal{L}(s_1 -_S s_2) = \mathcal{L}(s_1) -_T \mathcal{L}(s_2)$

(2) $\mathcal{L}(s_2) -_T \mathcal{L}(s_2) = 0_T$

It will be nice if S, T have "scalar multiplication"

(i.e. $(3f)(n) = 3\,f(n)$), and $\mathcal{L}$ respects scalar mult.

Wrote $(f_1 + f_2)(n) = f_1(n) + f_2(n)$

OR

$$f_1 + f_2 = f_1 -_S \underbrace{\left((-1) f_2\right)}_{\text{scalar mult in } S}$$

$\uparrow$

$+$ in S

A <u>real</u> vector space is a set V (S, T before)

s.t. (1) addition: $v_1, v_2 \in V$ then there is

$v_1 + v_2$ Jänich: $+ : V \times V \to V$

(2) scalar mult: $v_1 \in V$, $\alpha \in \mathbb{R}$, there is

αv_1 Jänich: $\cdot : \mathbb{R} \times V \to V$

s.t.

$$3(4v) = 12v$$

$$\alpha(\beta v) = (\alpha\beta)v \quad \forall \alpha, \beta \in \mathbb{R}, v \in V$$

A complex vector space

$\mathbb{R} \rightsquigarrow \mathbb{C}$

Complex vector space example:

Functions$(\mathbb{Z} \to \mathbb{C})$

Vector space (over $\mathbb{R}$, over $\mathbb{C}$ Jänich over $\mathbb{F}$)

field

Subspaces:

$\ker(\mathcal{L})$, $\mathcal{L}: \overbrace{\text{Functions}(\mathbb{Z} \to \mathbb{R})}^{S} \to$ same

$$(\mathcal{L}f)(n) = f(n+2) - f(n+1) - f(n)$$

$$\underbrace{\ker(\mathcal{L})}_{\text{subspace}} \subset S$$

Jan 23:

Examples of Vector Spaces:

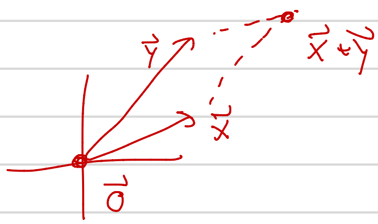
- Any set with (1) scalar multiplication over $\mathbb{R}$ (the reals)
  (2) an addition $+$ with $-$
  } satisfying a bunch of reasonable rules

<u>Ex</u>: $\mathbb{R}^n = \{ (x_1, \dots, x_n) \mid x_1, \dots, x_n \in \mathbb{R} \}$

we also write $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ "column vector"

$\vec{x} = (x_1, \dots, x_n)$, $\vec{y} = (y_1, \dots, y_n)$, $\vec{x}+\vec{y} = (x_1+y_1, \dots, x_n+y_n)$

"math"



"physics" sum of forces $\vec{F} = m\vec{a}$

$$\lambda (x_1, \dots, x_n) = (\lambda x_1, \dots, \lambda x_n)$$

Ex: Any set D, consider $\bar{V}$ = Functions$(D \to \mathbb{R})$

$f_1, f_2 \in \bar{V}$

$$(f_1 + f_2)(d) = f_1(d) + f_2(d)$$

<u>Textbook</u>: $D = [-1, 1]$ i.e. M = Functions$([-1,1] \to \mathbb{R})$

(1) Funct$((-1,1) \to \mathbb{R})$

(2) Differentiable Functs: $(-1, 1) \to \mathbb{R}$

(3) Infinitely " " : " " "

$\leq$ Subspaces

$$\mathrm{Poly}_{\leq 3}(\mathbb{R}) = \left\{ \begin{array}{c} \text{"formal expressions"} \\ a_0 + a_1 x + a_2 x^2 + a_3 x^3 \end{array} \middle| \; a_0, a_1, a_2, a_3 \in \mathbb{R} \right\}$$

$$\updownarrow$$

$$\{ (a_0, a_1, a_2, a_3) \mid a_0, a_1, a_2, a_3 \in \mathbb{R} \} = \mathbb{R}^4$$

$$\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$$

$$\mathrm{Poly}_{\leq 3}(\mathbb{R}) \subset \mathrm{Poly}_{\leq 7}(\mathbb{R}) \subset \mathrm{Poly}_{\substack{\text{any}\\ \text{degree}}}(\mathbb{R})$$

subsets

Sub <u>spaces</u>

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Formally: If $S \subset V$, S is a subset of V,

S is a subspace (sub-vector-space) if

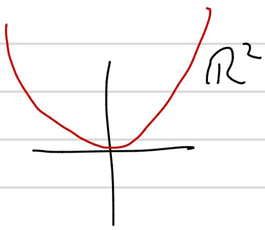
(1) ① if $\vec{x}, \vec{y} \in S$ then $\vec{x} + \vec{y} \in S$ $\quad$ $\vec{x}+\vec{y}$ (in V) <u>must</u> be in S

② if $\vec{x} \in S$, $\lambda \in \mathbb{R}$, then $\lambda \vec{x} \in S$

(2) S is a subspace of V if $S \overset{\text{subset}}{\subset} V$ and the $+$, scaler mult $(\cdot)$ in V give S the structure of a vector space

Typical exercise:

dim 2

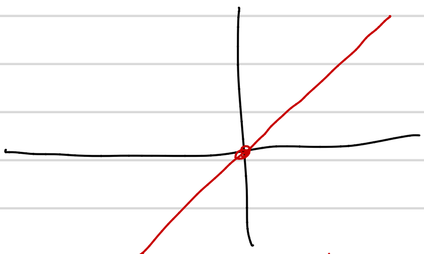


$$S = \{ (x,y) \mid y = x^2 \}$$

is this a <u>subspace</u>?

Subspaces of $\mathbb{R}^2$

line through $\vec{0}$



(dim 1)

just $\{\vec{0}\}$

(dim 0)

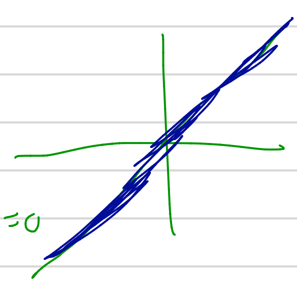
Not subspaces

$x - y = -3$

Subspaces have to contain $\vec{0}_{\mathbb{R}^2}$

(or subspace of $\vec{V}$ has to contain $\vec{0}_{\vec{V}}$)

but



$x - y = 0$

like $\mathbb{R}^4$

$\mathcal{P}_3 = \mathrm{Poly}_{\leq 3}(\mathbb{R})$:

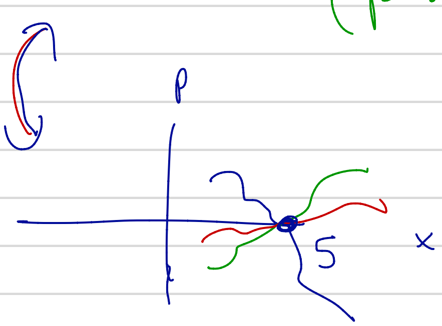
- is $\{ p \in \mathcal{P}_3 \mid p(5) = 0 \}$ a subspace?

- is $\{ p \in \mathcal{P}_3 \mid p(1) = p(2) = p(5) = 0 \}$ a subspace?

$$\Big\{ \underbrace{a_0 + a_1 x + a_2 x^2 + a_3 x^3}_{\text{"formal"}} \;\Big|\; \underbrace{a_0 + a_1 5 + a_2 5^2 + a_3 5^3 = 0}_{\text{"}p(5)=0\text{"}} \Big\}$$

Formally $p(5) = 0$, $q(5) = 0$. Is

$$(p+q)(5) \stackrel{?}{=} 0 \quad ?? \qquad \text{Yes}$$



is a subspace.

Is $S = \{ p \in \mathcal{P}_3 \mid p(5) = 2019 \}$ as subspace?

- is 0_{poly} in S NO!

- if $p, q \in S$, is $p + q \in S$

NO!

Jan 25 :

"Physicist" ?

§2.6 Dot Product
§1 of article

$\left.\begin{array}{l}\text{§2.6 Dot Product}\\ \text{§1 of article}\end{array}\right\}$ one topic

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Today: Ch 3 ; Homework #3 - do Problem 5 in LaTeX

" #4, 5, -- - do all problems in LaTeX

(Want to the The TeXbook by Knuth)

LaTeX ← "Family Sedan"

TeX ← "Sports Car"

Or : www.overleaf.com

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Dimension (Ch 3, Jänich)

where vector space come from

$\mathcal{L} : S \to T$

⇝ $\ker(\mathcal{L})$, ← uniqueness of eq. in $\mathcal{L}$

$\text{Image}(\mathcal{L})$ ← existence

HW #3

$\mathcal{L} : S \to T$, and S, T real vector spaces, then $\mathcal{L}$ is a <u>linear</u> transformation if

① $\mathcal{L}(s_1 + s_2) = \mathcal{L}(s_1) + \mathcal{L}(s_2)$

② $\mathcal{L}(\alpha s_1) = \alpha \mathcal{L}(s_1)$

$(\forall s_1, s_2 \in S, \ \alpha \in \mathbb{R})$

OR

(1+2) $\mathcal{L}(\alpha s_1 + \beta s_2)$

$= \alpha \mathcal{L}(s_1) + \beta \mathcal{L}(s_2)$

$(\forall s_1, s_2 \in S, \ \alpha, \beta \in \mathbb{R})$

"field"

textbook Jänich

Dimension:

If V is a $\left\{\begin{matrix}\text{complex} \\ \text{real}\end{matrix}\right\}$ $\mathbb{F}$ vector space, we say that $(v_1, \dots, v_n) \in V$ are <u>linearly independent</u> if the only $(\alpha_1, \dots, \alpha_n) \in \mathbb{R}$ for which

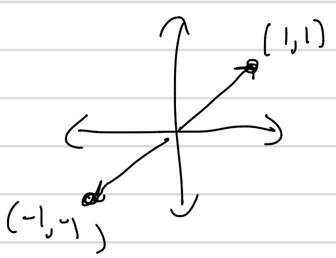
$$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = 0$$

is

$$(\alpha_1, \dots, \alpha_n) = (0, 0, \dots 0)$$

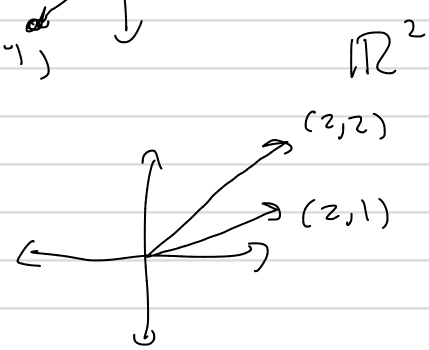
i.e.

$$\alpha_1 = 0$$
$$\alpha_2 = 0$$
$$\vdots$$
$$\alpha_n = 0$$



$$(1,1) + (-1,-1) = (0,0) = 0_{\mathbb{R}^2}$$

so $(1,1)$ and $(-1,-1)$ are dependent



are independent:

$$\alpha_1 (2,2) + \alpha_2 (2,1) = (0,0)$$

$$(2\alpha_1 + 2\alpha_2 \, , \, 2\alpha_1 + \alpha_2) = (0,0)$$

$$\begin{matrix} 2\alpha_1 + 2\alpha_2 = 0 \\ 2\alpha_1 + \alpha_2 = 0 \end{matrix} \implies \alpha_1 = \alpha_2 = 0$$

$\mathbb{R}^5$:

(1) $\left(\vec{0}, \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \\ 5 \end{bmatrix} \right)$ are dependent

$$5 \cdot \vec{0} + 0 \cdot v_2 + 0 v_3 + \dots + 0 v_n = 0$$

(2) $\left(\vec{1} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right)$ independent

$$\begin{bmatrix} \alpha_1 \\ \alpha_1 \\ \alpha_1 \\ \alpha_1 \\ \alpha_1 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\alpha_1 = 0$
$\alpha_1 = 0$
$\alpha_1 = 0 \quad \Rightarrow \quad \alpha_1 = 0$
$\alpha_1 = 0$
$\alpha_1 = 0$

(3) $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \vec{e}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \vec{e}_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

Standard "basis vectors" for $\mathbb{R}^5$

$$\alpha_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \vec{e}_2 + \dots + \alpha_5 \vec{e}_5 = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0_{\mathbb{R}^5} = \vec{0}$$

$$\Rightarrow \alpha_1 = \dots = \alpha_5 = 0$$

④ $\begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix}, \begin{bmatrix}5\\4\\3\\2\\1\end{bmatrix}$ lin indep

but

$$\begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix}, \begin{bmatrix}5\\4\\3\\2\\1\end{bmatrix}, \begin{bmatrix}2\\2\\2\\2\\2\end{bmatrix}$$

$$1 \cdot \quad + 1 \cdot \quad + (-3) \quad = \vec{0}$$

$\mathrm{Poly}_{\leq 3}(\mathbb{R})$: $1, x, x^2, x^3$ lin ind ?

$$\{ a_0 + a_1 x + a_2 x^2 + a_3 x^3 \mid a_0, \dots, a_3 \in \mathbb{R} \}$$

$V_1 = 1 = 1 + 0 \cdot x + 0 \cdot x^2 + 0 \cdot x^3$ $\longleftrightarrow$ Remind us of $\vec{e}_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$

$V_2 = x = 0 + 1 \cdot x + 0 \cdot x^2 + 0 \cdot x^3$

$V_3 = x^2 = 0 + 0 \cdot x + 1 \cdot x^2 + 0 \cdot x^3$ $\vdots$

$V_4 = x^3 = 0 + 0 \cdot x + 0 \cdot x^2 + 1 \cdot x^3$ $\vec{e}_4 = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$

Can

$$\alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3 + \alpha_4 V_4 = 0_{P_3}$$

$$\alpha_1 1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3 = 0 \text{ polynomial}$$

$\text{Poly}_{\leq 3}(\mathbb{R})$ $\qquad \binom{X}{0}, \binom{X}{1}, \binom{X}{2}, \binom{X}{3}$ independent?

$$\alpha_1 \binom{X}{0} + \alpha_2 \binom{X}{1} + \alpha_3 \binom{X}{2} + \alpha_4 \binom{X}{3} = 0$$

$$\alpha_1 \, 1 + \alpha_2 X + \alpha_3 \frac{X(X-1)}{2} + \alpha_4 \frac{X(X-1)(X-2)}{2\cdot 3} = 0_{\text{poly}}$$

Does this imply $\alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0, \alpha_4 = 0$

$X = 0$: $\quad \alpha_1 \cdot 1 + \alpha_2 \cdot 0 + \ldots = 0$

$$\alpha_1 = 0$$

Now: $\quad \alpha_2 X + \alpha_3 \frac{X(X-1)}{2} + \alpha_4 \frac{X(X-1)(X-2)}{2\cdot 3} = 0$

$X = 1$

$\alpha_2 = 0$

$$\alpha_3 \frac{X(X-1)}{2} + \alpha_4 \frac{X(X-1)(X-2)}{6} = 0$$

$X = 2$

$\alpha_3 = 0$

$\alpha_4 = 0$

Jan 28:

Last problem in TeX / LaTeX, overleaf.com is relatively easy

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Also (thanks to M.E. ← (initials)) detexify. etc is a good resource

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Chapter 3: "Basis" for a vector space

"Dimension" of " " "

"Linear independence" "

"Spanning (linear hull)" " " "

Say we work with $Poly_{\leq 3}(\mathbb{R})$.

$$= \left\{ \underbrace{\begin{matrix}\text{formal expressions} \\ a_0 + a_1 x + a_2 x^2 + a_3 x^3\end{matrix}} \;\middle|\; a_0, \dots, a_3 \in \mathbb{R} \right\}$$

think of it as $\rightsquigarrow (a_0, a_1, a_2, a_3) \in \mathbb{R}^4$

Say we have: $v_0 = 1, v_1 = x, v_2 = x^2, v_3 = x^3 \in V = Poly_{\leq 3}(\mathbb{R})$

Any element of $Poly_{\leq 3}(\mathbb{R})$ can be uniquely as

$$\alpha_0 v_0 + \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3, \qquad \alpha_0, \dots, \alpha_3 \in \mathbb{R}$$

A <u>basis</u> for a vector space, V, is a

(1) maximal set of <u>linearly</u> <u>independent</u> vectors in V

(2) a set of linearly independent vectors whose span is V

(2') tuple

(2'')

$(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n)$

$\mathrm{Span}(\vec{v}_1, \dots, \vec{v}_n)$

Convex Hull $(\vec{v}_1, \dots, \vec{v}_n)$

$$\overset{\text{def}}{=} \left\{ \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \dots + \alpha_n \vec{v}_n \;\middle|\; \alpha_1, \dots, \alpha_n \in \mathbb{R} \right\}$$

Thm: If V is spanned by some finite set of vectors, then all bases have the same size; this size is called the dimension of V.

E.g. $1, x, x^2, x^3$ is a basis for $\mathrm{Poly}_{\leq 3}(\mathbb{R})$;

dimension of $\mathrm{Poly}_{\leq 3}(\mathbb{R})$ is 4.

Say: $1+2+\dots+n$

$1+2^2+\dots+n^2$

$\vdots$

We like

$\binom{x}{0}, \binom{x}{1}, \binom{x}{2}, \binom{x}{3}$, i.e.

$1, \; x, \; \frac{x(x-1)}{2}, \; \frac{x(x-1)(x-2)}{6}$

Upshot: if $\binom{x}{0}$ $\binom{x}{1}$ $\binom{x}{2}$ $\binom{x}{3}$ are lin ind in V

($Poly_{\leq 3}(\mathbb{R})$)

and $\dim(V) = 4$, then any element of V

can be written as $\alpha_0 \binom{x}{0} + \alpha_1 \binom{x}{1} + \alpha_2 \binom{x}{2} + \alpha_3 \binom{x}{3}$.

e.g.

$$\sum_{i=1}^{n} p(n), \text{ where } p \text{ poly } \cancel{\text{deg}} \leq 3$$

$$p(x) = x^2+2, \quad \binom{x}{3} + x^2 + 5\binom{x}{2} + 7, \quad \ldots$$

How do we prove that $\binom{x}{0}, \binom{x}{1}, \ldots \binom{x}{3}$ are lin indep? Say

$$\alpha_0 \binom{x}{0} + \alpha_1 \binom{x}{1} + \alpha_2 \binom{x}{2} + \alpha_3 \binom{x}{3} = 0$$

$$\alpha_0 \cdot 1 + \alpha_1 \cdot x + \alpha_2 \frac{x(x-1)}{2} + \alpha_3 \frac{x(x-1)(x-2)}{6} = 0$$

took $x=0$

$$\Downarrow \; Eval_{x=0}$$

Abstractly:

$$Poly_{\leq 3}(\mathbb{R}) \xrightarrow{\text{linear trans}} \mathbb{R}$$

$$p \xmapsto[Eval_{x=0}]{} p(0)$$

$$\underbrace{\alpha_0 Ev_0(1)}_{1} + \underbrace{\alpha_1 Ev_0(x)}_{0} + \underbrace{\alpha_2 Ev_0(x^2)}_{0} + \underbrace{\alpha_3 Ev_0(x^3)}_{0} = 0$$

$$\alpha_0 \cdot 1 + 0 + 0 + 0 = 0$$

Jan 30:

Facts about linear independence, bases, dimension, etc.

(1) If $\{\vec{v}_1, \dots, \vec{v}_m\}$ are linearly independent vectors in $\mathbb{V}$ and $\text{Span}(\vec{v}_1, \dots, \vec{v}_m)$ is not all of $\mathbb{V}$, and $\vec{v} \notin \text{Span}(\vec{v}_1, \dots, \vec{v}_m)$ then $\{\vec{v}_1, \dots, \vec{v}_m, \vec{v}\}$ are linearly independent in $\mathbb{V}$.

So what? Ans: Every vector space has a basis!

(2) If $\{\vec{v}_1, \dots, \vec{v}_n\}$ is a basis for $\mathbb{V}$, and $\{\vec{u}_1, \dots, \vec{u}_m\}$ are linearly independent in $\mathbb{V}$, then $\mathbb{V}$ has a basis consisting of $\{\vec{u}_1, \dots, \vec{u}_m\}$ and some subset of $\{\vec{v}_1, \dots, \vec{v}_n\}$

So what? Dimension!

(3) If $\mathbb{V} = \text{Span}(\vec{v}_1, \dots, \vec{v}_p)$, then some subset of $\{\vec{v}_1, \dots, \vec{v}_p\}$ is a basis for $\mathbb{V}$.

So what? Simple notion of <u>finite</u> dimensional vector space.

Examples: $\mathbb{R}^n$, $\text{Poly}_{\leq n}(\mathbb{R})$, $\text{Functions}(\mathbb{Z} \to \mathbb{R})$, $\ker(\mathcal{L}_{Fib})$, $\text{Image}\big((x,y) \mapsto (x+2y, 2x+4y)\big)$, $\text{Image}(d/dx)$, $\ker(d/dx)$, $\text{Image}(\mathcal{D})$, $\ker(\mathcal{D})$

More in Ch. 3: $\dim(V_1+V_2) + \dim(V_1 \cap V_2) = \dim(V_1) + \dim(V_2)$

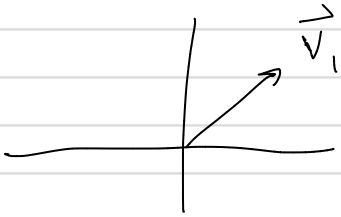
§3.4: We do by example

§3.5: Skip for now (cross prod)

§3.6: Historical (skip, except for some snide remarks about Pythagoras, Einstein, Marić-Einstein, Lobachevsky, etc.)

Examples of principles of Ch. 3 (dimension, basis, independence, span)

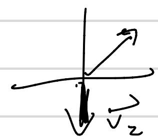
$\mathbb{R}^2$:



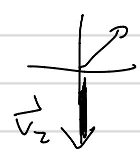
basis for $\mathbb{R}^2$: $(1,1)$ and ? , i.e. what $\vec{v_2}$ s.t. any element of $\mathbb{R}^2$ can be written uniquely as

$$\alpha_1 (1,1) + \alpha_2 \vec{v_2} \qquad ??$$

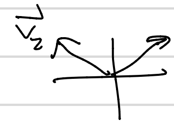
$\vec{v_2}$ can be ... $(0,-1)$



$(0,-2)$



$(-1,1)$



anything in $\mathbb{R}^2 \setminus \operatorname{span}(1,1)$

set difference

$$\{(x,y) \mid (x,y) \notin \operatorname{span}(1,1), \text{ i.e. } (x,y) \neq \alpha_1(1,1) \text{ for any } \alpha_1 \in \mathbb{R}\}$$

If $\vec{v}_1, \dots, \vec{v}_m$ are lin ind in V and $\vec{v} \notin \operatorname{span}(\vec{v}_1, \dots, \vec{v}_m)$, then

$$\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \dots + \alpha_m \vec{v}_m + \alpha \vec{v} = \vec{0}$$

then ....
- $\alpha \neq 0$ or else ... $\vec{v} \in \operatorname{span}(\dots)$
- but $\alpha = 0 \Rightarrow \alpha_1 \vec{v}_1 + \dots + \alpha_m \vec{v}_m = 0 \Rightarrow \alpha_i = 0$

$$\alpha_1 = \alpha_2 = \dots = \alpha_m = \alpha = 0$$

Functions$(\mathbb{Z} \to \mathbb{R})$

build a basis $= \{ \overset{\delta_1}{f_1}, \overset{\delta_2}{f_2}, \delta_3, \delta_4, \delta_5, \ldots$

$f_1 =$

$$f_1(n) = \begin{cases} 1 & \text{if } n=1 \\ 0 & \text{otherwise} \end{cases} \qquad \delta_2(n) = \begin{cases} 1 & \text{if } n=2 \\ 0 & \text{otherwise} \end{cases}$$

$\delta_1(n)$

$=$

If $\alpha_1\delta_1 + \alpha_2\delta_2 = 0$

$(\alpha_1\delta_1 + \alpha_2\delta_2)(1)$

$= \alpha_1\delta_1(1) + \alpha_2\delta_2(1)$

$= \alpha_1$

$=$

$$\text{Functions}(\mathbb{Z} \to \mathbb{R}) \supset \text{span}\overbrace{(\delta_1, \delta_2, \delta_3, \ldots \delta_{10^{10^{10}}}, \ldots)}^{\text{linearly independent basis ?}}$$

$\delta_{-1} \notin$

When building bases, adding linearly indep vectors one-by-one, but never stop, say V is infinite dimensional.

What about $f(n) = \begin{cases} 0 & n \le 5 \\ n & n=6 \end{cases}$

$\ker(\mathcal{L}_{Fib})$

$$\mathcal{L}_{Fib} : \text{Functions}(\mathbb{Z} \to \mathbb{R}) \to \text{Function}(\mathbb{Z} \to \mathbb{R})$$

$\ker(\mathcal{L}_{Fib}) =$

$$V = \left\{ f : \mathbb{Z} \to \mathbb{R} \;\middle|\; \begin{array}{c} f(n+2) - f(n+1) - f(n) = 0 \\ \text{for all } n \in \mathbb{Z} \end{array} \right\}$$

$\dim V \stackrel{?}{=} 2$:

Given $f(0), f(1)$ determines $f(2), f(3), \ldots$
$f(-1), f(-2), \ldots$

Favourite bases:

$$\left\{ f_0 : \begin{array}{l} f(0) = 1 \\ f(1) = 0 \end{array}, \quad f_1 : \begin{array}{l} f(0) = 0 \\ f(1) = 1 \end{array} \right\}$$

f_0: , $f_0(0) = 1$, $f_0(1) = 0$, $f_0(2) = 1$, $f_0(3) =$

$\ldots, 2, -1, 1, 0, 1, 1, 2, 3, 5, 8, 13, \ldots$

f_1: $\ldots, -1, 1, 0, 1, 1, 2, 3, \ldots$

If we know

$$f(0) = 1984, \ f(1) = 2019$$

and $f \in \bar{V}$, then

$$f = 1984 \, f_0 + 2019 \, f_1$$

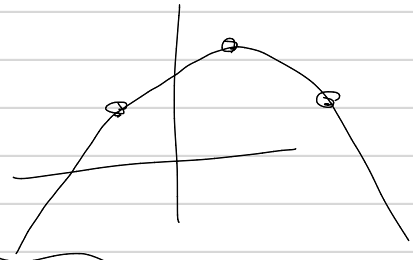
Favourite basis:

$g(n)$: ζ_+^n : $\ldots, \overset{g(0)}{\zeta_+^{-1}}, \overset{g(1)}{1}, \zeta_+, \zeta_+^2, \ldots$

$\tilde{g}(n)$: ζ_-^n : $\ldots, \zeta_-^{-1}, 1, \zeta_-, \zeta_-^2$

Feb 1

§2 of article:



curve fitting!

$y = a_2 x^2 + a_1 x + a_0$

here: data points

a_2, a_1, a_0 unknown

Need: any 3x3 system

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

(1) always has a unique solution for fixed a_{ij} coefficients regardless of b_1, b_2, b_3

OR

(2) never has a unique solution - - - - - -

\- — - — -

=

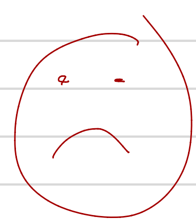
Today we prove this:

$$0 \cdot x_1 + 1 \cdot x_2 + 2 \cdot x_3 = b_1$$

$$\frac{2019}{2019} \cdot x_1 + \frac{1984}{2019} \cdot x_2 + \frac{5}{2019} x_3 = \frac{b_2}{2019}$$

$$2019 \cdot x_1 + 1986 \cdot x_2 + 7 x_3 = b_3$$

. . . .



$$0\cdot x_1 + 1\cdot x_2 + 2\cdot x_3 = b_1$$

$$\frac{2019}{2019}\cdot x_1 + \frac{1984}{2019}\cdot x_2 + \frac{5}{2019} x_3 = \frac{b_2}{2019}$$

$$2019\cdot x_1 + 1986\cdot x_2 + 7 x_3 = b_3 \quad \ldots$$

=

① What is the dimension of $\mathbb{R}^3$, real 3-dim space?

3

Def: $\vec{v}_1, \ldots, \vec{v}_n \in V$ vector space are a <u>basis</u> for V

if $\vec{v}_1, \ldots, \vec{v}_n$ are lin indep and

$$\left\{ \begin{array}{l} \mathrm{Span}(\vec{v}_1, \ldots, \vec{v}_n) \\ L(\vec{v}_1, \ldots, \vec{v}_n) \end{array} \right\} = V$$

E.g. $\vec{e}_1 = (1,0,0)$, $\vec{e}_2 = (0,1,0)$, $\vec{e}_3 = (0,0,1)$ are a basis for $\mathbb{R}^3$.

Rem: For a basis, every vector in V can be written uniquely as a linear combination of basis elements.

Why? If

$$\vec{v} = \begin{cases} \alpha_1 \vec{v_1} + \dots + \alpha_n \vec{v_n} \\ \beta_1 \vec{v_1} + \dots + \beta_n \vec{v_n} \end{cases}$$

$$\vec{0} = (\alpha_1 - \beta_1)\vec{v_1} + \dots + (\alpha_n - \beta_n)\vec{v_n}$$

Upshot:

① Every basis for V has the same size;

"basis exchange lemma"

Definition: This size is called the dimension of V

② If $\dim(V) = d$, then

- all bases have size d.

$\mathbb{R}^3$ = bases $\vec{e_1}, \vec{e_2}, \vec{e_3}$

$(1,1,1), (1,2,4), (1,3,9)$ basis

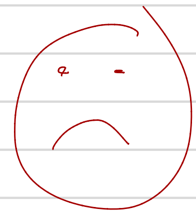
$(-5,3,2), (2019, 1, 17.3), (-1984, 2, 5)$

- any d linearly independent vectors of V form a basis (i.e. ~~they~~ they span all of $\mathbb{R}^3$)
- if d vectors don't span all of V, i.e. so $\mathrm{Span}(v_1, \dots, v_d)$ is a proper subspace of V, then $v_1, \dots, v_d$ lin dep

$$0 \cdot x_1 + 1 \cdot x_2 + 2 \cdot x_3 = b_1$$

$$2019 \cdot x_1 + 1984 \cdot x_2 + 5 \, x_3 = b_2$$

$$2019 \cdot x_1 + 1986 \cdot x_2 + 7 \, x_3 = b_3$$



look at

$$\begin{bmatrix} 0 \\ 2019 \\ 2019 \end{bmatrix}, \begin{bmatrix} 1 \\ 1984 \\ 1986 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix}$$

~~either~~ → is a basis

→ isn't

If is a basis, then any $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ can be written as a combination

$$\alpha_1 \begin{bmatrix} 0 \\ 2019 \\ 2019 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 1984 \\ 1986 \end{bmatrix} + \alpha_3 \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix}$$

$$\Longleftarrow$$

$$\alpha_1 0 + \alpha_2 1 + \alpha_3 2 = b_1$$

$$\alpha_1 2019 + \alpha_2 1984 + \alpha_3 5 = b_2$$

$$\alpha_1 2019 + \alpha_2 1986 + \alpha_3 7 = b_3$$

$$\begin{bmatrix} 0 \\ 2019 \\ 2019 \end{bmatrix}, \begin{bmatrix} 1 \\ 1984 \\ 1986 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix}$$

~~either~~ → ~~is a basis~~

→ isn't

$$\alpha_1 \begin{bmatrix} 0 \\ 2019 \\ 2019 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 1984 \\ 1986 \end{bmatrix} + \alpha_3 \begin{bmatrix} 2 \\ 7 \\ 5 \end{bmatrix} = \vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Feb 4:

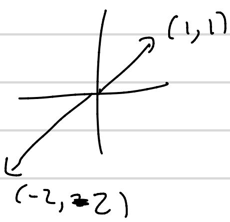
3 vectors in $\mathbb{R}^3$:

① They are a <u>basis</u>

You actually have to $\left\{ \begin{array}{l} \text{reasoning} \\ \text{matrix calculation} \end{array} \right\}$ to tell ① vs ②

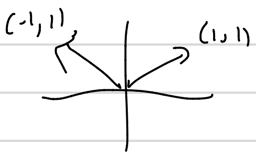
② They are <u>not</u> a basis

In $\mathbb{R}^2$



lin dependent ← $\begin{bmatrix} 1 \\ 1 \end{bmatrix} x_1 + \begin{bmatrix} -2 \\ -2 \end{bmatrix} x_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

has ∞ many solutions



- independent
- span all of $\mathbb{R}^2$
- basis
- $\det \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \neq 0$
- $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ invertible
- ⋮

← $\begin{bmatrix} 1 \\ 1 \end{bmatrix} x_1 + \begin{bmatrix} -1 \\ 1 \end{bmatrix} x_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

has a unique solution

=

- Basis exchange example

example: basis: $\underset{v_1}{1}, \underset{v_2}{x}, \underset{v_3}{x^2}, \underset{v_4}{x^3}$ for $Poly_{\leq 3}(\mathbb{R}) = P_3 = \vec{V}$

vectors $\binom{x}{2} = \dfrac{x(x-1)}{2}$.

$\dim(P_3) = 4$: basis $\vec{v}_1 = 1$, $\vec{v}_2 = x$, $\vec{v}_3 = x^2$, $\vec{v}_4 = x^3$

new vector $\binom{x}{2} = \vec{w}_1$

$$\vec{w}_1 = \frac{x(x-1)}{2} = \frac{1}{2}x^2 - \frac{1}{2}x = \frac{1}{2}\vec{v}_3 - \frac{1}{2}\vec{v}_2$$

V = basis: $\vec{v}_1 = 1$, $\vec{v}_2 = x$, $\vec{v}_3 = x^2$, $\vec{v}_4 = x^3$

$$\vec{w}_1 = \binom{x}{2} = \frac{x(x-1)}{2} = \frac{1}{2}x^2 - \frac{1}{2}x = \frac{1}{2}\vec{v}_3 - \frac{1}{2}\vec{v}_2$$

$$\vec{w}_1 = \frac{1}{2}\vec{v}_3 - \frac{1}{2}\vec{v}_2$$

"basis exchange"

$$\frac{1}{2}\vec{v}_2 = \frac{1}{2}\vec{v}_3 - \vec{w}_1$$

$$\vec{v}_2 = \vec{v}_3 - 2\vec{w}_1$$

$\vec{v}_1, \cancel{\vec{v}_2}, \vec{v}_3, \vec{v}_4, \vec{w}_1$ work?

$$\text{Span}(\vec{v}_1, \quad, \vec{v}_3, \vec{v}_4, \vec{w}_1) = \mathcal{P}_3 = \text{Poly}_{\leq 3}(\mathbb{R})$$

$\Rightarrow$ $\vec{v}_1, \vec{v}_3, \vec{v}_4, \vec{w}_1$ is a basis

one basis to another

$$7x^2 + 3x + 1 = 1 \cdot \vec{v}_1 + 3 \cdot \vec{v}_2 + 7 \cdot \vec{v}_3$$

$$= 1 \cdot \vec{v}_1 + 3(\vec{v}_3 - 2\vec{w}_1) + 7\vec{v}_3$$

Example 2:

$\vec{v}_1 = 1, \ \vec{v}_2 = x, \ \vec{v}_3 = x^2, \ \vec{v}_4 = x^3$

$V = P_3 = Poly_{\le 3}(\mathbb{R})$

$\boxed{\vec{w}_1 = \binom{x}{2}, \quad \vec{w}_2 = \binom{x}{3}}$ lin indep

bring in $\vec{w}_1$, out goes some $\vec{v}_i$, $i = 1, \dots, 4$

(in our case $\vec{v}_2$ out)

Basis: $\vec{v}_1, \vec{v}_3, \vec{v}_4, \vec{w}_1$

Next: $\vec{w}_2$ in, out $\underbrace{\vec{v}_1, \vec{v}_3, \vec{v}_4}_{\text{one of these leaves}}, \underbrace{\vec{w}_1}_{\text{keep}}$

$$\vec{w}_2 = \_\,\vec{v}_1 + \_\,\vec{v}_3 + \_\,\vec{v}_4 + \_\,\vec{w}_1$$

?? $0 \cdot \vec{v}_1 + 0\,\vec{v}_3 + 0\,\vec{v}_4 + z\,\vec{w}_1$

forced to move $\vec{w}_1$ in ???

Feb 6

- Homework has a lot of short explanations.
- Office hours this week

Wed 3:30-5pm (me), MA 1118

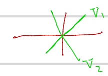
Thu 2-3pm (me), Math Building room 210

- Focus: $\bar{V}_1, \bar{V}_2 \subset \bar{V}$ subspaces:

$$\dim(V_1 \cap V_2) + \dim(V_1 + V_2)$$

$$= \dim(V_1) + \dim(V_2)$$

("modularity" of dim).

$\mathbb{R}^2$: 

$\bar{V}_1 + \bar{V}_2 := \{ \vec{v}_1 + \vec{v}_2 \in \bar{V} \mid \vec{v}_1 \in \bar{V}_1, \vec{v}_2 \in \bar{V}_2 \}$

$= \mathrm{Span}(\bar{V}_1 + \bar{V}_2)$

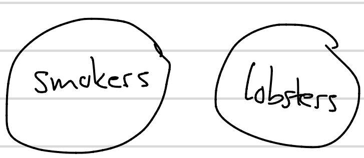
Here $\bar{V}_1 + \bar{V}_2 = \mathbb{R}^2$

$\bar{V}_1 \cap \bar{V}_2 = \{0\}$

$\dim(\bar{V}_1 + \bar{V}_2) = 2$ $\quad$ $\dim(\bar{V}_1) = 1$

$\dim(\bar{V}_1 \cap \bar{V}_2) = 0$ $\quad$ $\dim(\bar{V}_2) = 1$

- Examples in $\mathbb{R}^2, \mathbb{R}^3$:
- Homework: Sometimes $\dim(V_1), \dim(V_2), \dim(V_1 \cap V_2)$ easy to determine, but $\dim(V_1 + V_2)$ not so obvious...
- Analogue:



A_1 $\quad$ A_2

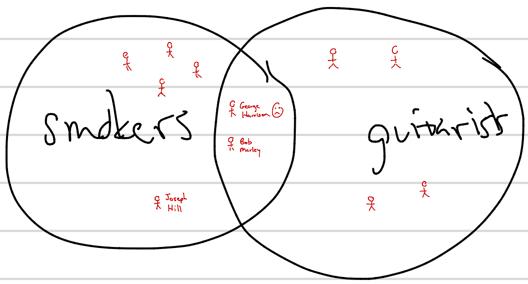
$A_1 \cap A_2 = \emptyset$

$|A_1| + |A_2| = |A_1 \cup A_2|$

$|A_1 \cup A_2|$

$= |A_1| + |A_2|$

$- |A_1 \cap A_2|$



$$|S \cup G| = |S| + |G| - |S \cap G|$$

4x6 Chocolate Bar Example

16x20 " " "

[Next page]

$$\dim(\bar{V}_S + \bar{V}_G) = \dim(\bar{V}_S) + \dim(\bar{V}_G) - \dim(\bar{V}_S \cap \bar{V}_G)$$

$A = \{1, 2, \dots, 7\}$, $A_1 = \{1, 2, 3\}$, $A_2 = \{3, 4, 5, 6\}$

$|A_1| = 3$
$|A_2| = 4$
$|A_1 \cap A_2| = 1$
$|A_1 \cup A_2| = 6$

$$\mathbb{R}^{A_1} = \text{Functions}(\{1,2,3\} \to \mathbb{R}) = \mathbb{R}^{\{1,2,3\}}$$

(view as $\mathbb{R}^3 = \{(a_1, a_2, a_3) \mid a_1, \dots \in \mathbb{R}\}$)

subspace of Functions$(\{1, 2, 3, \dots, 7\} \to \mathbb{R})$

via "extension by zero"

f: $f(1) = 2019$
$f(2) = 1984$
$f(3) = 1986$

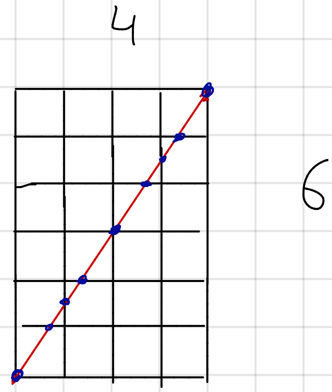
extend by zero

$f(1) = 2019$
$f(2) = 1984$
$f(3) = 1986$
$f(4) = 0$
$f(5) = 0$
$f(6) = 0$
$f(7) = 0$

4×6 Chocolate Bar

12×16 " "

etc.



9 blue dots

7 horizontal (6+1)
5 vertical (4+1)
3 both (GCD(4,6) + 1)

$$= \quad \underbrace{\mathbb{R}^{\{1,2\}}}_{\text{Funct } \{1,2\}\to\mathbb{R}} \cap \underbrace{\mathbb{R}^{\{1,3\}}}_{\text{Funct } \{1,3\}\to\mathbb{R}} \overset{\text{maybe}}{\underset{??}{=}} \underbrace{\mathbb{R}^{\{1\}}}_{\text{Funct } \{1\}\to\mathbb{R}}$$

Want $\underset{\text{2-dim}}{\mathbb{R}^{\{1,2\}}}, \underset{\text{2-dim}}{\mathbb{R}^{\{1,3\}}}, \ldots \subset \mathbb{R}^{\{1,2,3\}}$

$\left\{ \begin{matrix} 1 \mapsto 71 \\ 2 \mapsto 2019 \end{matrix} \right\} = \mathbb{R}^{\{1,2\}}$ as function in $\mathbb{R}^{\{1,2,3\}}$

extend by zero:
$$\begin{aligned} 1 &\mapsto 71 \\ 2 &\mapsto 2019 \\ 3 &\mapsto 0 \end{aligned}$$

A_1	A_2	$A_1 \cap A_2$	$A_1 \cup A_2$
$\{1,2,3\}$	$\{3,4,5,6\}$	$\{3\}$	$\{1,2,3,4,5,6\}$

$|A_1 \cup A_2| = 6$

$$\underbrace{\dim\left(\mathbb{R}^{\{1,2,3\}}\right)}_{V_1} + \dim\left(\mathbb{R}^{\{3,4,5,6\}}\right)_{V_2} \quad \dim\left(\mathbb{R}^{\{3\}}\right)_{V_1 \cap V_2} + \dim\left(\mathbb{R}^{\{1,2,3,4,5,6\}}\right)_{V_1 + V_2}$$

dim 6

all viewed as subspaces of $\mathbb{R}^{\{1,2,\dots,7\}}$

(or $\mathbb{R}^{\{1,2,3,\dots,2019\}}$)

via "extension by zero"

=

$\mathbb{R}^{\{1,2\}}$ $\quad$ $\mathbb{R}^{\{1,2\}}$

V_1 $\quad$ V_2

$$\dim(V_1) + \dim(V_2) = \dim(V_1 \cap V_2) + \dim(V_1 + V_2)$$

$$2 + 2 = 2 + 2$$

$$4 = 4$$

Feb 8:

Homework 5:

2×2, 7×7 basis exchange
3×3, ...

$\begin{bmatrix}4\\4\end{bmatrix}, \begin{bmatrix}6\\5\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$

$v_1 \quad v_2 \quad \delta_1 \quad \delta_2$

$4x_1 + 6x_2 = b_1$
$4x_1 + 5x_2 = b_2$

$$\left.\begin{aligned} 4x_1 + 6x_2 + 0x_3 &= b_1 \\ 4x_1 + 5x_2 + 0x_3 &= b_2 \\ 0x_1 + 0x_2 + 7x_3 &= b_3 \end{aligned}\right\} \text{solve}$$

Later: systematic Gaussian elimination...

$\underset{\vec{v}_1}{\begin{bmatrix}4\\4\\0\end{bmatrix}} \quad \underset{\vec{v}_2}{\begin{bmatrix}6\\5\\0\end{bmatrix}} \quad \underset{\vec{v}_3}{\begin{bmatrix}0\\0\\7\end{bmatrix}} \qquad \underset{\vec{\delta}_1 = \vec{e}_1}{\begin{bmatrix}1\\0\\0\end{bmatrix}} \quad \underset{\vec{e}_2}{\begin{bmatrix}0\\1\\0\end{bmatrix}} \quad \underset{\vec{e}_3}{\begin{bmatrix}0\\0\\1\end{bmatrix}}$

$$\vec{v}_1 = 4\vec{e}_1 + 4\vec{e}_2 + 0\vec{e}_3$$
$$\vec{v}_2 = 6\vec{e}_1 + 5\vec{e}_2 + 0\vec{e}_3$$
$$\vec{v}_3 = 0\vec{e}_1 + 0\vec{e}_2 + 7\vec{e}_3$$

$$\vec{v}_1 = 4\vec{e}_1 + 4\vec{e}_2 + 0\vec{e}_3$$
$$\vec{v}_2 = 6\vec{e}_1 + 5\vec{e}_2 + 0\vec{e}_3$$
$$\vec{v}_3 = 0\vec{e}_1 + 0\vec{e}_2 + 7\vec{e}_3$$

$$\vec{v}_1, \vec{v}_2, \vec{v}_3 \xrightarrow{\text{dictionary}} \vec{e}_1 \quad \vec{e}_2 \quad \vec{e}_3$$

goal

$$\vec{e}_1, \vec{e}_2, \vec{e}_3 \xrightarrow{\text{dictionary}} \vec{v}_1, \vec{v}_2, \vec{v}_3$$

$$\left.\begin{aligned}\vec{v}_3 &= 7\vec{e}_3 \\ \tfrac{1}{7}\vec{v}_3 &= \vec{e}_3 \\ \vec{e}_3 &= \tfrac{1}{7}\vec{v}_3\end{aligned}\right\} \text{basis pivot/exchange}$$

$$\vec{v}_1, \vec{v}_2, \vec{e}_3 \qquad \vec{e}_1, \vec{e}_2 \quad \vec{v}_3$$

$$\vec{v}_1 = 4\vec{e}_1 + 4\vec{e}_2 + 0\vec{e}_3$$
$$\vec{v}_2 = 6\vec{e}_1 + 5\vec{e}_2 + 0\vec{e}_3$$
$$\cancel{\vec{v}_3 = 0\vec{e}_1 + 0\vec{e}_2 + 7\vec{e}_3}$$
$$\vec{e}_3 = \tfrac{1}{7}\vec{v}_3$$

$\vec{v}_1, \vec{v}_2, \vec{v}_3 \quad \vec{e}_1, \vec{e}_2, \vec{e}_3$

$\vec{v}_1 = 4\vec{e}_1 + 4\vec{e}_2 + 0\vec{e}_3$

$\vec{v}_2 = 6\vec{e}_1 + 5\vec{e}_2 + 0\vec{e}_3$

$\vec{e}_3 = \frac{1}{7}\vec{v}_3$

$\vec{v}_1, \vec{v}_2, \vec{e}_3 \quad \vec{e}_1, \vec{e}_2, \vec{v}_3$

$\vec{v}_1 = 4\vec{e}_1 + 4\vec{e}_2 \; : \; \vec{e}_1 = \frac{1}{4}\vec{v}_1 - \vec{e}_2$

$\vec{e}_1, \vec{v}_2, \vec{e}_3 \quad \vec{v}_1, \vec{e}_2, \vec{v}_3$

$\vec{e}_3 = \frac{1}{7}\vec{v}_3$

$\vec{e}_1 = \frac{1}{4}\vec{v}_1 - \vec{e}_2$

$\vec{v}_2 = \frac{6}{4}\vec{v}_1 - \vec{e}_2 + 0\vec{v}_3$

$6\vec{e}_1 + 5\vec{e}_2$

$6\left(\frac{1}{4}\vec{v}_1 - \vec{e}_2\right) + 5\vec{e}_2$

$\vec{e}_3, \vec{e}_1, \vec{v}_2 \quad \vec{v}_1, \vec{e}_2, \vec{v}_3$

$$\vec{e}_3 = \frac{1}{7}\vec{v}_3$$

$$\vec{e}_1 = \frac{1}{4}\vec{v}_1 - \vec{e}_2$$

$$\boxed{\vec{v}_2 = \frac{6}{4}\vec{v}_1 - \vec{e}_2} + 0\,\vec{v}_3$$

$$\vec{e}_3, \vec{e}_1, \vec{v}_2 \qquad \vec{v}_1, \vec{e}_2, \qquad \vec{v}_3$$

$$\vec{e}_3, \vec{e}_1, \vec{e}_2 \qquad \vec{v}_1, \vec{v}_2, \vec{v}_3$$

$$\vec{e}_2 = \frac{6}{4}\vec{v}_1 - \vec{v}_2 + 0$$

$$\vec{e}_1 = \frac{1}{4}\vec{v}_1 - \vec{e}_2$$

$$= \frac{1}{4}\vec{v}_1 - \left(\frac{6}{4}\vec{v}_1 - \vec{v}_2\right)$$

$$= \frac{-5}{4}\vec{v}_1 + \vec{v}_2$$