Math 223

Notes starting Jan 2, 2019

Math 223, Jan 2, 2019

(Projector near windows looks yellow)

- Welcome to 2019
- Honours linear algebra
- Resources
  - Online Textbook
  - Short online article (first 2 weeks or so)
    (return to this during term)
    (prereq: high school lin algebra, 2×2 system
  - Some calculations in
    - Julia ← (In LSK 310) also free
    - MATLAB ← You have in math labs (LSK 310)

- Friday: Survey:
  - Which applications you prefer
  -

Grade: $(.55)f + (.35)\max(f, m) + (.10)\max(f, m, h)$

f = final, m = midterm, h = homework

Start with article: start with Section 3:

$$1+2+\dots+n = \frac{n(n+1)}{2} = p_1(n)$$

$$1+2^2+\dots+n^2 = \frac{n(n+1)(2n+1)}{6} = p_2(n)$$

$$1+2^3+\dots+n^3 = \frac{n^2(n+1)^2}{4} = p_3(n)$$

etc. $\qquad = (1+2+\dots+n)^2$ ☺

Use to illustrate
- Summation notation
- Proofs by induction
- Backwards sequences

} Section 0

- Change of basis... ← Cover later in Math 223; fundamental idea

$$1^2+2^2+\dots+n^2 = \sum_{m=1}^{n} m^2$$

$a, b \in \mathbb{Z} = \{\dots -2, -1, 0, 1, 2, \dots\}$ ← integers

$\mathbb{N} = \{1, 2, 3, \dots\}$ natural numbers

$\mathbb{R}$ = real numbers $\qquad$ $\mathbb{C}$ = complex numbers

Use $\mathbb{C}$ later

the set of (all)

$$\mathbb{C} = \{ a + ib \mid a, b \in \mathbb{R} \}, \quad i = \text{“}\sqrt{-1}\text{”}$$

formal sum

such that

e.g. $3+2i$, $7-i$, $\pi + i e^{(\pi^{2.1})}$, ... $\qquad \pi = 3.14159\ldots$

$a, b \in \mathbb{Z}$, $a \le b$, $\qquad f$ function on integers

$$\sum_{m=a}^{m=b} f(m) = f(a) + f(a+1) + \ldots + f(b)$$

e.g. $\displaystyle\sum_{m=1}^{5} m^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 =$ whatever

capital Greek Σ, σ, ς

$$\sum_{m=1}^{1} m^2 = 1^2 = 1$$

$$\sum_{m=1}^{0} m^2 = 0 \qquad \text{(why?)}$$

$$\sum_{m=1}^{-3} m^2 =$$

$$\sum_{m=1}^{2} m^2 = 1^2 + 2^2 \qquad = \sum_{m=1}^{3} m^2 - 3^2$$

$$\sum_{m=1}^{1} m^2 = 1^2 \qquad = \left(\sum_{m=1}^{2} m^2\right) - 2^2$$

$$\sum_{m=1}^{0} m^2 \overset{?}{=} 0 = \left(\sum_{m=1}^{1} m^2\right) - 1^2$$

$$\sum_{m=1}^{-1} m^2 = 0 \leftarrow \left(\sum_{m=1}^{0} m^2\right) - 0^2$$

$$\sum_{m=1}^{-2} m^2 = -1 \qquad \left(\sum_{m=1}^{-1} m^2\right) - (-1)^2$$

if $a > b$

$$\sum_{m=a}^{b} f(m) = -f(b+1) - \cdots - f(a-1)$$

More practically:

$$p_2(n) := 1 + 2^2 + 3^2 + \ldots + n^2 = \text{poly of deg } 3$$

$p_2(-2),\ p_2(-1),\ p_2(0),\ p_2(1),\ p_2(2),\ p_2(3),\ p_2(4)$

$\ldots, -1,\ 0,\ 0,\ 1,\ 1+4=5,\ 1+4+9=14,\ 30$

$\ldots, -1 \xrightarrow{+(-1)^2} 0 \xrightarrow{+0} 0 \xrightarrow{+1} 1 \xrightarrow{+4} 5 \xrightarrow{+9} 14 \xrightarrow{+16} 30$

The sequence $1, 5, 14, 30, 30+5^2, 30+5^2+6^2, \ldots$

extends backwards

$\ldots, -14, -5, -1, 0, 0, 1, 5, 14, 30, 55,$

$p_2(-1)$ $p_2(0)$

What is $p_2(n)$

$p_2(x)$ deg 3 poly in x

roots: when is $p_2(x) = 0$?

$x = 0$

$x = -1$

$x = -1/2$

$\ldots, -14, -5, -1 \quad 0 \;;\; 0, \quad 1, \quad 5, \quad 14, \ldots$

$p_2(-2) \quad p_2(-1) \;;\; p_2(0) \quad p_2(1)$

For any $n \in \mathbb{Z}$

$$\boxed{p_3(n) = -p_3(-1-n)}$$

$$p_3(0) = -p_3(-1)$$

$$p_3(1) = -p_3(-1-1)$$

What is $\underbrace{p_3(x) + p_3(-1-x)}_{0} = ?$

Next time

$$p_3(-1/2) = 0$$

$$p_3(x) = x(x+1)\left(x+\tfrac{1}{2}\right) \cdot C$$

$$= x(x+1)\frac{(2x+1)}{2} \cdot C'$$

Jan 4:

Last time: grade: $(.55)f + (.35)\max(f, m) + (.10)\max(f, m, h)$

=

2-3 week give applications of linear algebra

=

Looked at

$$\underbrace{1 + 2^2 + 3^2 + \dots + n^2}_{\sum_{m=1}^{n} m^2} = \frac{n(n+1)(2n+1)}{6}$$

$\leftarrow$ bi-infinite sequence

=

$$S_1 = 1, \quad S_2 = 1 + 2^2 = 5, \quad S_3 = \underbrace{1 + 2^2}_{S_2} + 3^2 = S_2 + 9 = 14, \dots$$

$$S_1 = 1, \quad \text{and} \quad S_{n+1} = S_n + (n+1)^2$$

$$\underset{S_1}{1}, \underset{S_2}{5}, \underset{S_3}{14}, \underset{S_4}{30}, \underset{S_5}{55}, \dots$$

backwards

$$S_n = S_{n+1} - (n+1)^2$$

$$S_0 = S_1 - 1^2 = 0$$

$$S_{-1} = S_0 - 0^2 = 0$$

$$S_{-2} = S_{-1} - (-1)^2 = -1, \quad S_{-3} = S_{-2} - (-2)^2 = -5, \quad -14, \dots$$

outside the box

$$\underbrace{S_{-3} \quad S_{-2} \quad S_{-1} \quad S_0}_{\text{outside the box}} \quad \boxed{S_1,\ S_2,\ S_3,\ S_4 \ \ 1,\ 5,\ 14,\ 30, \ldots}$$

$-5,\ -1,\ 0,\ 0,$

box

Assume: $\sum_{m=1}^{n} m^2 = $ poly deg 3 in n $= p_2(n) = p(n)$

If so, $p(0)=0$

$p(-1)=0$

Also ... $p(-1/2)=0$

$\Rightarrow p(x)$ deg 3

$$p(x) = \underbrace{\begin{pmatrix}\text{leading}\\ \text{coef}\end{pmatrix}}_{1/3} x\left(x+\tfrac{1}{2}\right)(x+1)$$

in a few minutes

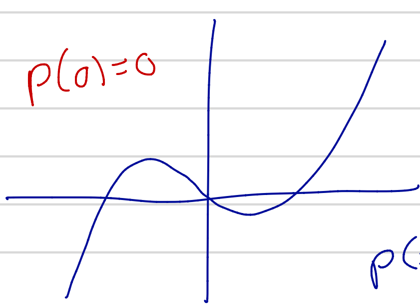
$$\left(\Leftarrow \int_{x=0}^{x=t} x^2\,dx = t^3/3 \right)$$

$$p(x) = \frac{1}{3}\, x\,(x+1)\left(x+\tfrac{1}{2}\right) = \frac{x(x+1)(2x+1)}{6}$$

$=$

What is an $\left\{\begin{matrix}\text{odd}\\ \text{even}\end{matrix}\right\}$ polynomial?

(homework)



$p(0)=0$

odd

$p(-x) = -p(x)$

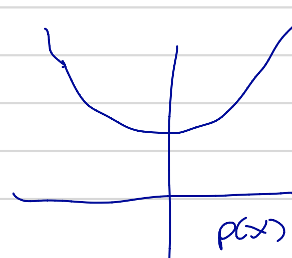
$p(0) = -p(0)$

$p(x) = a_1 x + a_3 x^3 + \ldots$

odd powers

even

$p(x) = p(-x)$



$p(x) = a_0 x^0 + a_2 x^2 + \ldots$

even powers

$-5 \quad -1 \quad 0 \;\vdots\; 0 \quad 1, 5, 14, 30$ ↙ polynomial

$\quad\quad\quad S_{-1} \;\vdots\; S_0 \quad S_1 \quad S_2 \; \ldots$

$p_2(x)$ shift by $-1/2$ is odd polynomial

Point! If p is a poly of degree at most d, and $p(x)=0$ for $d+1$ values of x, then $\underbrace{p=0}_{}$.

$p =$ zero polynomial

$$p_2(-x) = -p_2(-1+x)$$

$$q(x) := \underbrace{p_2(-x)}_{\text{deg 3 poly}} + \underbrace{p_2(-1+x)}_{\text{deg 3 poly}} = \text{poly of degree at most 3}$$

$$\boxed{}$$

$$\binom{1}{2} + \binom{2}{2} + \binom{3}{2} + \ldots + \binom{n}{2} = \binom{n+1}{3}$$

$$\frac{n(n-1)}{2} = \frac{(n+1)(n)(n-1)}{6}$$

Jan 7:

Section 3 of article on applications:

$$1+2+\dots+n = \frac{n(n+1)}{2}$$

$$1^2+2^2+\dots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\vdots$$

⇝ Abstract vector space, change of basis

Maybe $\boxed{\binom{1}{2}+\binom{2}{2}+\binom{3}{2}+\dots+\binom{n}{2} = \binom{n+1}{3}}$?

Is this true? $\binom{n}{2} = \frac{n(n-1)}{2}$, $\binom{n+1}{3} = \frac{(n+1)(n)(n-1)}{2\cdot 3}$

Define: $\binom{n}{k} = \frac{n(n-1)\dots(n-k+1)}{1\cdot 2\cdot 3\cdot\dots\cdot k} = \frac{n!}{k!\,(n-k)!}$

$k! = 1\cdot 2\cdot\dots\cdot k$, $(n-k)! = 1\cdot 2\cdot\dots\cdot(n-k)$

$n! = 1\cdot 2\cdot\dots\ (n-1)\,n$

For us: $\binom{m}{1} = m$, $\binom{m}{2} = \frac{m(m-1)}{2}$ $\binom{m}{3} = \frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}$

$$\boxed{\binom{1}{2} + \binom{2}{2} + \binom{3}{2} + \dots + \binom{n}{2} = \binom{n+1}{3}}$$

$$\boxed{\binom{n+1}{3} - \binom{n}{3}} = \frac{(n+1)\,n\,(n-1)}{2\cdot 3} - \frac{n(n-1)(n-2)}{2\cdot 3}$$

$$= n(n-1)\left(\frac{n+1}{2\cdot 3} - \frac{n-2}{2\cdot 3}\right)$$

$$= n(n-1)\,\frac{1}{2} = \boxed{\binom{n}{2}}$$

$$\binom{6}{3} - \binom{5}{3} = \binom{5}{2}$$

$$\binom{5}{3} - \binom{4}{3} = \binom{4}{2}$$

$$\binom{4}{3} - \binom{3}{3} = \binom{3}{2}$$

$$\binom{3}{3} - \binom{2}{3} = \binom{2}{2}$$

$$\binom{2}{3} - \binom{1}{3} = \binom{1}{2}$$

$$\binom{6}{3} - \underbrace{\binom{1}{3}}_{0} = \binom{1}{2} + \binom{2}{2} + \dots \binom{5}{2}$$

$$\binom{6}{3} =$$

In general: $\binom{n+1}{3} - \binom{n}{3} = \binom{n}{2}$

$$\binom{m}{2} = \frac{m(m-1)}{2} = \frac{1}{2}m^2 - \frac{m}{2}$$

$$\rightsquigarrow \binom{1}{2} + \binom{2}{2} + \dots + \binom{n}{2} = \binom{n+1}{3}$$

$$\sum_{m=1}^{n} \binom{m}{2} = \binom{n+1}{3}$$

$$= \sum_{m=1}^{n} \left(\frac{1}{2}m^2 - \frac{1}{2}m\right)$$

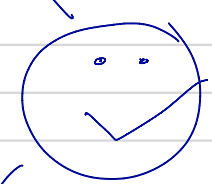
$$= \sum_{m=1}^{n} \frac{1}{2}m^2 - \sum_{m=1}^{n} \frac{1}{2}m = \binom{n+1}{3} = \frac{(n+1)(n)(n-1)}{2\cdot 3}$$

$$= \frac{1}{2}\left(\sum_{m=1}^{n} m^2\right) - \frac{1}{2}\left(\sum_{m=1}^{n} m\right)$$

$$= \frac{1}{2}\underbrace{\left(1+2^2+\dots+n^2\right)}_{\text{Also poly}} - \frac{1}{2}\underbrace{\left(1+2+\dots+n\right)}_{\frac{n(n+1)}{2}} = \frac{(n+1)n(n-1)}{2\cdot 3}$$

poly

poly

$\frac{1}{2}$  $\left(\frac{n^3}{3} + \text{poly}_{\deg \le 2}\right)$ $- \frac{1}{2}\left(\text{order } n^2 + \text{lower}\right) = \frac{n^3}{6} + n^2 \text{ term} + \dots$

Next are:

$$\binom{1}{3} + \binom{2}{3} + \cdots + \binom{n}{3} = \binom{n+1}{4} \leftarrow \text{poly deg } 4$$

$$\frac{n^4}{4!}$$

$$\sum_{m=1}^{n} \binom{m}{3}$$

$$= \sum_{m=1}^{n} \frac{m(m-1)(m-2)}{6}$$

$$= \sum_{m=1}^{n} \left[\frac{m^3}{6} + m^2\left(\frac{-3}{6}\right) + m\left(\frac{2}{6}\right) \right]$$

$$= \frac{1}{6}(1+2^3+\cdots+n^3) + \left(\frac{-3}{6}\right)(1+\cancel{2}2^2+\cdots+n^2) + \left(\frac{2}{6}\right)(1+2+\cdots+n)$$

$=$

Idea: took $\binom{n}{3}, \binom{n+1}{3}$ difference $\binom{n+1}{3} - \binom{n}{3} = \binom{n}{2}$:)

i.e. $f(n) = \binom{n}{3}$

$\rightsquigarrow$ $f(n+1) - f(n) \rightarrow$ new function

Define: If $f: \mathbb{N} \to \mathbb{N}$ (or similar), $\mathcal{D}f$ is the ~~function~~ function

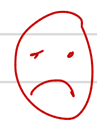
$$\boxed{(\mathcal{D}f)(n)} = f(n+1) - f(n)$$

Example: $f(n) = \binom{n}{3}$, $\mathcal{D}f$ is the function:

$$(\mathcal{D}f)(n) = \binom{n+1}{3} - \binom{n}{3} = \binom{n}{2}$$

Ex. 2 $f(n) = n^3$

$$(\mathcal{D}f)(n) = (n+1)^3 - n^3 =$$



$$= 3n^2 + 3n + 1$$

Ex 3: $f(n) = \binom{n}{12}$

$$(\mathcal{D}f)(n) = \binom{n}{11}$$

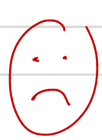
$n-11$?

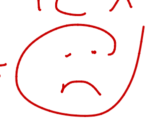
$$\binom{n+1}{12} - \binom{n}{12} = \frac{(n+1)(n)\cdots(n-10)}{11!\cdot 12} - \frac{n(n-1)\cdots(n-11)}{11!\cdot 12}$$

$$= \frac{n(n-1)\cdots(n-10)}{11!}\cdot\frac{(n+1)-(n-11)}{12} = \binom{n}{11}$$

Ex 4 $f(n) = n^{12}$

$$(\mathcal{D}f)(n) = (n+1)^{12} - n^{12} =$$

Compare: $\frac{d}{dx}\binom{x}{12} =$  $\qquad \frac{d}{dx}\left(x^{12}\right) = 12x^{11}$

$\mathcal{D}$ " $\binom{x}{11}$ $\qquad \mathcal{D}\left(x^{12}\right) =$ 

Jan 9:

Last time defined $\mathcal{D}$ "difference operator"

taking function f and returning a new function

$\mathcal{D}f$ given by:

$$(\mathcal{D}f)(n) := f(n+1) - f(n).$$

Here f can be $\mathbb{Z} \to \mathbb{Z}$ or $\mathbb{R} \to \mathbb{R}$ or $\left\{ \begin{array}{l} - \ \mathbb{Z} \to \mathbb{R} \\ - \text{ polynomial in } n \\ - \text{ continuous funct of } x \in \mathbb{R} \\ - \text{ etc.} \\ - \text{ etc.} \\ - \text{ etc.} \end{array} \right\}$

but not $\left\{ \begin{array}{l} \mathbb{N} \to \mathbb{N}, \text{ e.g.} \\ f(1) = 5,\ f(2) = 4, \text{ so} \\ (\mathcal{D}f)(1) = f(2) - f(1) \\ \qquad = 4 - 5 = -1 \end{array} \right\}$

Ex: $f: \mathbb{N} \to \mathbb{N}$ is strictly increasing, is $\mathcal{D}f$ the same?

Trick $\quad \mathcal{D}$ "$\binom{n}{3}$" $=$ "$\binom{n}{2}$"

(there is only one variable, n, so $\mathcal{D}$ is understood)

hence $\quad \binom{1}{2} + \binom{2}{2} + \dots + \binom{n}{2} = \binom{n+1}{3} - \binom{1}{3}$

abstractly: for any f

$$\underbrace{(\mathcal{D}f)(1)}_{f(2) - f(1)} + \underbrace{(\mathcal{D}f)(2)}_{f(3) - f(2)} + \dots + \underbrace{(\mathcal{D}f)(n)}_{f(n+1) - f(n)} = f(n+1) - f(1)$$

Another operator $\mathcal{S}$ taking f to

$$(\mathcal{S} f)(n) := f(1) + f(2) + \dots + f(n)$$

$$\text{i.e. } \sum_{m=1}^{n} f(m)$$

Here we want domain of f to be $\mathbb{N}, \mathbb{Z}$.

Abstract theorem:

$$(\mathcal{D}f)(1) + (\mathcal{D}f)(2) + \dots + (\mathcal{D}f)(n) = f(n+1) - f(1),$$

i.e.

$$\boxed{\text{FOR ANY } f : \mathbb{Z} \to \left\{ \begin{matrix} \mathbb{Z} \\ \mathbb{R} \\ \text{poly} \\ \text{etc.} \end{matrix} \right\} \quad (\mathcal{S}(\mathcal{D}f))(n) = f(n+1) - f(1)}$$

We want p = poly s.t.

$$(\mathcal{D}p)(n) = n^2$$

$$p = \frac{(n-1)(n)(2n-1)}{6} = \left.\frac{m(m+1)(2m+1)}{6}\right|_{m=n+1}$$

$$p(n+1) = \frac{n(n+1)(2n+1)}{6}$$

$$p(1) = 0$$

$$\Rightarrow \quad 1 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} - 0$$

$$p(n) = \frac{(n-1)n(2n-1)}{6} + C$$

has $(\mathcal{D}p)(n) = n^2$

C has to be 0

$p(1) = 1$

To solve

$$(\mathcal{D}p)(n) = \text{given function of } n$$

and 2 solutions

$$\mathcal{D}p_1 = \mathcal{D}p_2 =$$

then

$$\mathcal{D}(p_1 - p_2) = \text{zero function}$$

Thm: $\mathcal{D}f = \text{zero funct} \Rightarrow f(2) - f(1) = 0$, $f(3) - f(2) = 0$, $\vdots$ $\Rightarrow f = \text{constant}$

Cor: $\mathcal{D}p_1 = \mathcal{D}p_2$ then $p_1 - p_2 = \text{Constant}$

So $p_2 = p_1 + \text{Constant}$

Jan 11

From Wikipedia page on Bernoulli numbers

... Atque si porrò ad altiores gradatim potestates pergere, levique negotio sequentem adornare laterculum licet :

Summae Potestatum

$\int n = \frac{1}{2}nn + \frac{1}{2}n$
$\int nn = \frac{1}{3}n^3 + \frac{1}{2}nn + \frac{1}{6}n$
$\int n^3 = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}nn$
$\int n^4 = \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n$
$\int n^5 = \frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}nn$
$\int n^6 = \frac{1}{7}n^7 + \frac{1}{2}n^6 + \frac{1}{2}n^5 - \frac{1}{6}n^3 + \frac{1}{42}n$
$\int n^7 = \frac{1}{8}n^8 + \frac{1}{2}n^7 + \frac{7}{12}n^6 - \frac{7}{24}n^4 + \frac{1}{12}nn$
$\int n^8 = \frac{1}{9}n^9 + \frac{1}{2}n^8 + \frac{2}{3}n^7 - \frac{7}{15}n^5 + \frac{2}{9}n^3 - \frac{1}{30}n$
$\int n^9 = \frac{1}{10}n^{10} + \frac{1}{2}n^9 + \frac{3}{4}n^8 - \frac{7}{10}n^6 + \frac{1}{2}n^4 - \frac{1}{12}nn$
$\int n^{10} = \frac{1}{11}n^{11} + \frac{1}{2}n^{10} + \frac{5}{6}n^9 - 1n^7 + 1n^5 - \frac{1}{2}n^3 + \frac{5}{66}n$

Quin imò qui legem progressionis inibi attentuis ensperexit, eundem etiam continuare poterit absque his ratiociniorum ambabimus : Sumtâ enim c pro potestatis cujuslibet exponente, fit summa omnium n^c seu

$$\int n^c = \frac{1}{c+1}n^{c+1} + \frac{1}{2}n^c + \frac{c}{2}An^{c-1} + \frac{c\cdot c-1\cdot c-2}{2\cdot 3\cdot 4}Bn^{c-3}$$
$$+\frac{c\cdot c-1\cdot c-2\cdot c-3\cdot c-4}{2\cdot 3\cdot 4\cdot 5\cdot 6}Cn^{c-5}$$
$$+\frac{c\cdot c-1\cdot c-2\cdot c-3\cdot c-4\cdot c-5\cdot c-6}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8}Dn^{c-7}\cdots \text{\& ita deinceps,}$$

exponentem potestatis ipsius n continué minuendo binario, quosque perveniatur ad n vel nn. Literae capitales A, B, C, D & c. ordine denotant coëfficientes ultimorum terminorum pro $\int nn$, $\int n^4$, $\int n^6$, $\int n^8$, & c. nempe

$$A = \frac{1}{6}, B = -\frac{1}{30}, C = \frac{1}{42}, D = -\frac{1}{30}.$$

Jan 11

Finish §3 today : "change of basis"

Do §2 Lin Alg w/o Lin Alg

Do §1 Curve Fitting (Linear Regression)

§7 Page Rank (involve MATLAB / Julia)

next week

=

Systematic Sum of Squares, Cubes, etc.

Review: $\boxed{f(n) = \binom{n}{3}}$

$$f(n+1) - f(n) = (\mathcal{D}f)(n) = \binom{n}{2}$$

$$\mathcal{D}\binom{n}{4} = \binom{n}{3}$$

$$\binom{1}{2} + \binom{2}{2} + \cdots \binom{n}{2} = \binom{n+1}{3} - \binom{1}{3}$$

$$n^2 \cdot \frac{1}{2} - n\left(\frac{1}{2}\right) = \frac{n(n-1)}{2}$$

expanded

$$\mathcal{S}\binom{n}{2} = \binom{n+1}{3} - \binom{1}{3}$$

$$(\mathcal{S}f)(n) := f(1) + f(2) + \cdots + f(n)$$

$$\underbrace{\binom{1}{3} + \dots + \binom{n}{3}} = \binom{n+1}{4} - \binom{1}{4}$$

=

Systematic : Philosophy

What is a poly deg 3 ?

Formal expression: $a_0 + a_1 n + a_2 n^2 + a_3 n^3$, $a_3 \neq 0$

$$\mathrm{Poly}_{\le 3}(\mathbb{R}) = \underset{\text{Textbook}}{P_3} = \left\{ \begin{array}{l} \text{formal expression} \\ a_0 + a_1 x + a_2 x^2 + a_3 x^3 \end{array} \,\middle|\, a_i \in \mathbb{R} \right\}$$

What about alternate def?

$$\mathrm{Poly}_{\le 3}(\mathbb{R}) = b_0\binom{n}{0} + b_1\binom{n}{1} + b_2\binom{n}{2} + b_3\binom{n}{3} \ ?$$

$\binom{n}{0} = 1$, $\binom{n}{1} = n$, $\binom{n}{2} = \frac{n(n-1)}{2} = \frac{1}{2}n^2 - \frac{1}{2}n$, $\binom{n}{3} = \frac{n(n-1)(n-2)}{6}$

$$b_0\binom{x}{0} + b_1\binom{x}{1} + b_2\binom{x}{2} + b_3\binom{x}{3}$$

$\binom{n}{2}$ = bn 2 th[illegible]

$= \frac{n(n-1)}{2}$

$$\binom{x}{0} = 1$$

$$\binom{x}{1} = x$$

$$\binom{x}{2} = -\frac{1}{2}x + \frac{1}{2}x^2$$

$$\frac{1}{2}x^2 = \binom{x}{2} + \frac{1}{2}x$$

$$= \binom{x}{2} + \frac{1}{2}\binom{x}{1}$$

$$1 = \binom{x}{0}$$

$$x = \binom{x}{1}$$

$$x^2 = \binom{x}{1} + 2\binom{x}{2} \quad \leftarrow \text{say}$$

$$\Delta(n^2) = \Delta\binom{n}{1} + 2\,\Delta\binom{n}{2}$$

$$\Delta\left(\binom{n}{1} + 2\binom{n}{2}\right)$$

$$\underbrace{\binom{n+1}{2}}_{\binom{n}{2}+\binom{n}{1}} + 2\cdot\binom{n+1}{3}$$

$$\binom{n}{2} + \binom{n}{1} + 2\left(\binom{n}{3} + \binom{n}{2}\right)$$

$\binom{n+1}{2} - \binom{n}{2}$

$\shortparallel$

$\Delta\binom{n}{2}$

$= \binom{n}{1}$

LUCK

§2 Lin Alg w/o Lin Alg!

$$\boxed{\begin{aligned} x + 2y &= 5 \\ 2x + 4y &= 33 \end{aligned}}$$

$\leftrightarrow$ $2x+4y=10$

no solution ☹

$$\boxed{\begin{aligned} x + 2y &= 5 \\ 2x + 4y &= 10 \end{aligned}}$$

infinitely many ☹

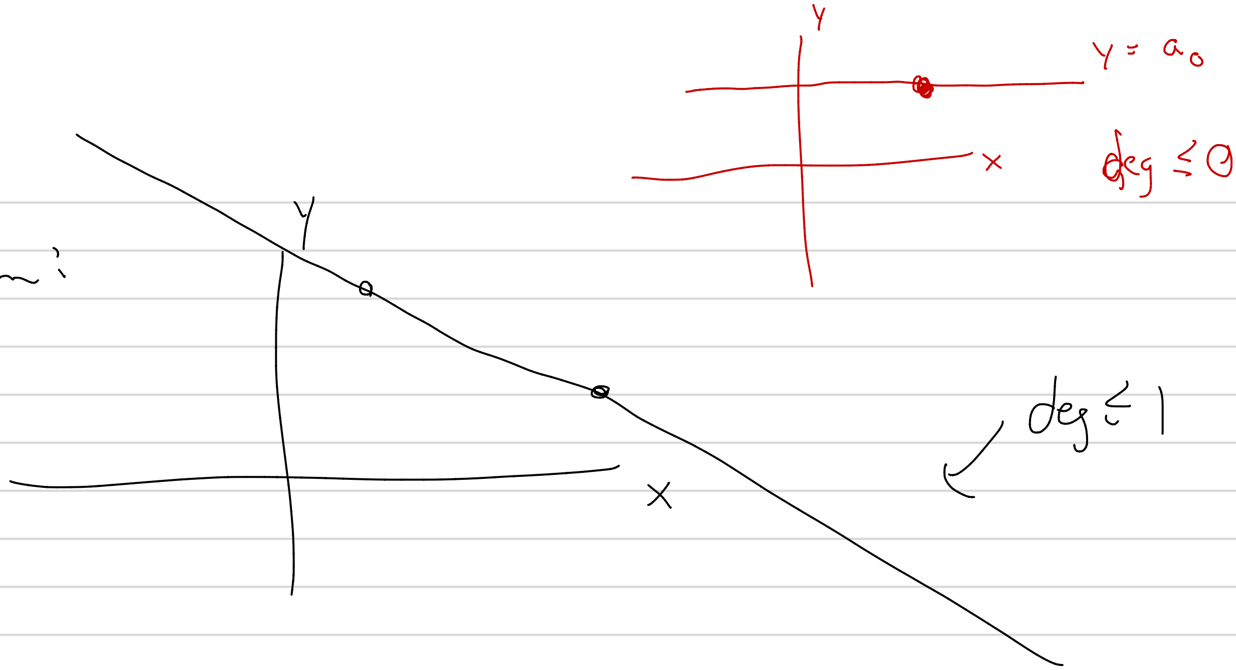
$x + 2y = \cancel{5}$ blah $\in \mathbb{R}$

$2x + 3y = \cancel{200}$ blahblah $\in \mathbb{R}$

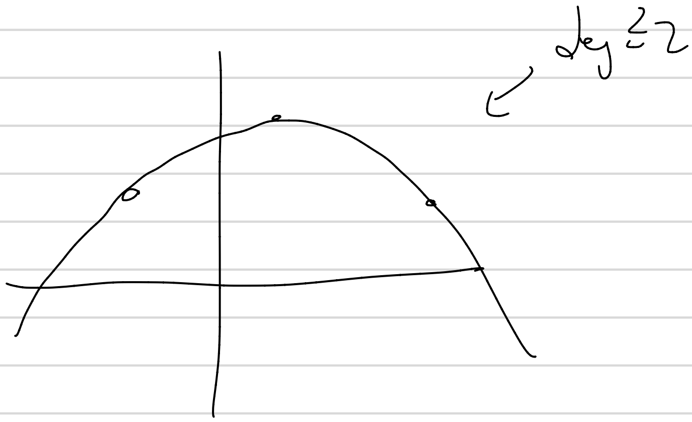
If you have n linear, in n variables $x_1, \dots, x_n$

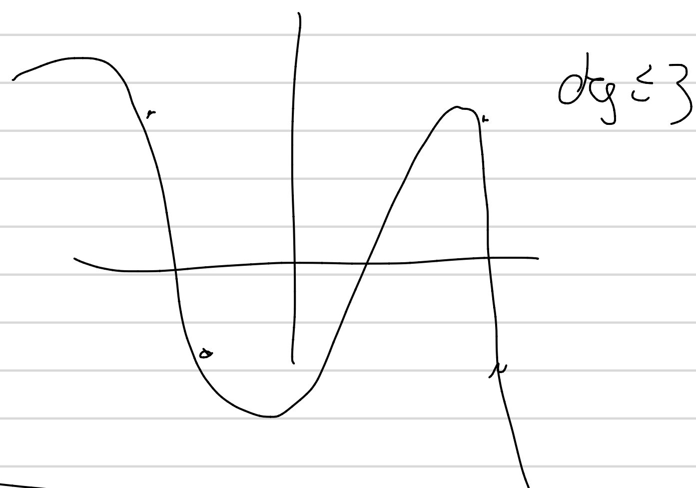
Real numbers $\{a_{ij}\}, \{b_i\}$

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ \vdots \qquad \vdots \qquad \ddots \qquad \vdots \quad &\quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n &= b_n \end{aligned}$$



Claim:





$$\boxed{\begin{array}{l}\text{If } p = p(x) \text{ degree } \le k \\ \text{and } p(x) = 0 \text{ has } \ge k+1 \text{ solutions,} \\ \text{then } p = 0 \text{ (polynomial)}\end{array}}$$

3 x's, 2 y's — fix this

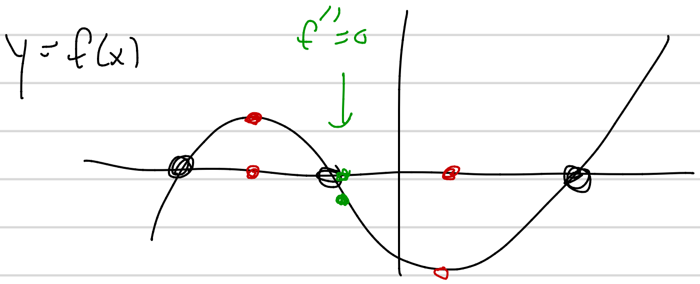
Jan 14:

Section 2 (then Section 1)

Fact: If $p(x) = a_0 + a_1 x + a_2 x^2$ and

$p(x) = 0$ for distinct $x_1 < x_2 < x_3$

then $p = 0$ polynomial, i.e. $a_0 = a_1 = a_2 = 0$



$f'(x) = 0$ has 2 distinct solutions

$f''(x) = 0$ has 1 solution

Rolle's theorem

Or:

$p(x)$ is divisible by $(x - x_1)(x - x_2)(x - x_3)$

Study:
Systems of 2 linear equations in 2 unknowns

Make claims " " n " " " n "

versus

$$x + 2y = \qquad\qquad x + 2y =$$
$$2x + 4y = \qquad\qquad x + 3y =$$

some constants   some constants

$$1 \cdot x + 2y = 1 \qquad\qquad x + 2y = 5$$
$$2x + 4y = 3 \qquad\qquad x + 3y = 7$$

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}}_{\text{coefficient}} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\text{variables}} = \underbrace{\begin{bmatrix} 1 \\ 3 \end{bmatrix}}_{\text{constants}} \qquad \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

"Homogeneous form"

$$1 \cdot x + 2y = 0 \qquad\qquad 1 \cdot x + 2 \cdot y = 0$$
$$2x + 4y = 0 \qquad\qquad 1 \cdot x + 3y = 0$$

infinitely many solutions   put zeros   unique solution (i.e. $x=0$, $y=0$) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ put 0's

Later: If have n equations, n unknowns

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n$$

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & \cdots & & a_{nn} \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

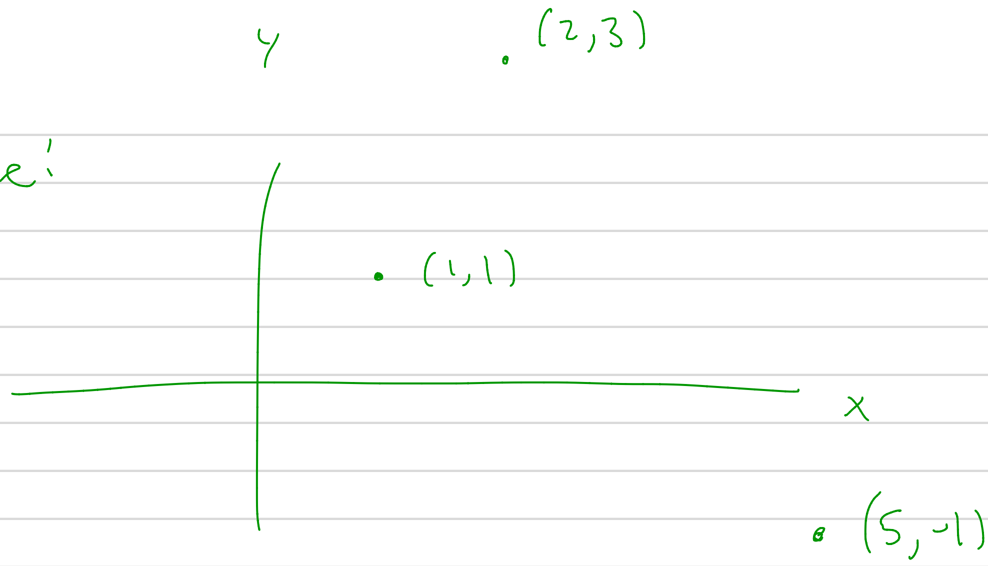
$$A\vec{x} = \vec{b}$$

then it has a unique solution iff the "homogeneous version"

$$A\vec{x} = \vec{0}$$

has a unique solution.

For example:



Are there (unique) a_0, a_1, a_2 s.t.

$$y = a_0 + a_1 x + a_2 x^2 \quad \text{for} \quad (x,y) = \begin{cases} (1,1) \\ (2,3) \\ (5,-1) \end{cases}$$

(1,1): $1 = a_0 + a_1 \cdot 1 + a_2 1^2$

(2,3): $3 = a_0 + a_1 \cdot 2 + a_2 2^2$

(5,-1): $-1 = a_0 + a_1 \cdot 5 + a_2 \cdot 5^2$

(variables are a_0, a_1, a_2)

homog:

$$\left.\begin{aligned} 0 &= a_0 + a_1 \cdot 1 + a_2 1^2 \\ 0 &= a_0 + a_1 \cdot 2 + a_2 2^2 \\ 0 &= a_0 + a_1 \cdot 5 + a_2 \cdot 5^2 \end{aligned}\right\}$$

poly $y = a_0 + a_1 x + a_2 x^2$ thru $\begin{cases} (1,0) \\ (2,0) \\ (5,0) \end{cases}$

but this has unique solution

= Same: deg 2 poly, 3 points $\rightsquigarrow$ deg k poly, k+1 points

Jan 16

§2 : If $n \times n$ system,

i.e. $n \times n$ linear system,

i.e. n equations for n unknowns (over $\mathbb{R}$ = reals, $\mathbb{C}$, $\mathbb{Q}$, ...)

$A\vec{x} = \vec{b}$ has a unique sol

iff

$A\vec{x} = \vec{0}$ has a unique sol

Homework: Uniqueness:

$$f(x) = f_1(x) + f_2(x)$$

subtract and

$$f(x) = f_3(x) + f_4(x)$$

where f_3 even, f_4 odd

g is even if $g(x) = g(-x)$

g is odd $g(x) = -g(-x)$

show $f_1 = f_3$, $f_2 = f_4$

$$0 = \underbrace{(f_1 - f_3)}_{\text{even}} + \underbrace{(f_2 - f_4)}_{\text{odd}}$$

This step makes things easier

$$A\vec{x} = \vec{b}$$

$$A\vec{x}' = \vec{b}$$

is $\vec{x} = \vec{x}'$ necessarily?

$$A(\vec{x} - \vec{x}') = \vec{b} - \vec{b} = \vec{0}$$

In other words:

$$A\vec{x} = A\vec{x}' \quad \text{iff} \quad A(\vec{x} - \vec{x}') = \vec{0}$$

$$\int x^2 = \frac{1}{3}X^3 + C \longleftarrow C = \text{constant}$$

$$\int \cos(x)\,dx = \sin(x) + C$$

Equation: $\frac{d}{dx}(f) = x^2$ $\qquad$ $\frac{d}{dx}(f) = \cos(x)$

Say $f = \frac{1}{3}x^3 + 10^{10^{10}}$ solves

$$\left.\begin{aligned} \frac{d}{dx}(f) &= x^2 \\ \frac{d}{dx}(g) &= x^2 \end{aligned}\right\} \quad \frac{d}{dx}(f - g) = 0$$

diff function $\quad$ diff function $\quad$ function

homog eq $\quad \boxed{\frac{d}{dx}(h) = 0} \iff h = C$ constant

$$\mathcal{L}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathcal{L}(f) = \frac{d}{dx} f$$

etc.

Abstractly

$$\mathcal{L} : S \to T$$

When does equation

$$\mathcal{L}(s) = \underbrace{\text{something given in } T}$$

have a unique solution?

difficult

Say $\mathcal{L}(s_1) = \mathcal{L}(s_2) =$

(1) subtraction in T

in T

(2) $\mathcal{L}(s_1) - \mathcal{L}(s_2)$ BIG ASSUMPTION !!! !!!!

$= \mathcal{L}(s_1 - s_2)$

(3) subtract in S

"subtract" $\mathcal{L}(s_2)$ both sides

$$\mathcal{L}(s_1) - \mathcal{L}(s_2) = \underbrace{\mathcal{L}(s_2) - \mathcal{L}(s_2)}$$

$$\mathcal{L}(\underbrace{s_1 - s_2}_{\text{lives in } S}) = 0 = 0_{\text{element in } T}$$

Upshot

$$\mathcal{L}(s_1) = \mathcal{L}(s_2) \overset{}{\underset{\text{Assumptions}}{\iff}} s_1 - s_2 \in \begin{array}{l}\text{Kernel}(\mathcal{L}) \\ \text{Nullspace}(\mathcal{L})\end{array}$$

$$\overset{\text{def}}{=} \{ s \in S \mid \mathcal{L}(s) = 0 \}$$

Jan 18:

Last time: $\mathcal{L} : S \to T$ make assumptions that $\mathcal{L}$ is <u>linear</u>

$$\mathcal{L}(s_1) = \mathcal{L}(s_2)$$

$$\iff \mathcal{L}(s_1 - s_2) = 0_T$$

$$\iff s_1 - s_2 \in \ker(\mathcal{L})$$

where $\ker(\mathcal{L}) = \{ s \in S \mid \mathcal{L}(s) = 0_T \}$

We needed: subtractions in S, T and need

$$\mathcal{L}(s_1 - s_2) = \mathcal{L}(s_1) - \mathcal{L}(s_2)$$

Example: $\mathcal{L} : \mathbb{R} \to \mathbb{R}$ s.t. $\mathcal{L}(x) = x^2$, then $\mathcal{L}$ is <u>not</u> linear

$$(3-1)^2 \neq 3^2 - 1^2$$

$$(a+b)^2 \neq a^2 + b^2 \quad \text{in general}$$

=

$\mathcal{L} : \mathbb{R} \to \mathbb{R}$, $\mathcal{L}(x) = 2019 \cdot x$, $\mathcal{L}$ is "linear"

$\mathcal{L} : \mathbb{R}^2 \to \mathbb{R}^2$, $\mathcal{L}(x,y) = (x+2y, 2x+4y)$

OR

$$\mathcal{L}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+2y \\ 2x+4y \end{bmatrix} \quad \text{linear.}$$

integers to reals

Say $\mathcal{L} : S \to T$, where $S = T = \text{Functions}(\mathbb{Z} \to \mathbb{R})$

We say $\mathcal{L}$ is <u>linear</u> if

① $$\mathcal{L}(s_1 - s_2) = \mathcal{L}(s_1) - \mathcal{L}(s_2)$$

② If $\alpha \in \mathbb{R}$, $s \in S$, then

$$\mathcal{L}(\alpha s) = \alpha \mathcal{L}(s).$$

=

Subtraction in $S = T = \text{Func}(\mathbb{Z} \to \mathbb{R})$: $\quad f_1, f_2 \in S$

$$(f_1 - f_2)(n) = f_1(n) - f_2(n)$$

Scaling / multiplication

$$(2019 f_1)(n) = 2019 f_1(n)$$

more generally

$$(\alpha f_1)(n) = \alpha f_1(n), \quad \alpha \in \mathbb{R}$$

$$f_1 + f_2 = f_1 \underset{\text{subtraction}}{-} \underbrace{(-1) f_2}_{\text{scale by } -1}$$

=

$\mathcal{L}$ is <u>linear</u> if

$(f_1, f_2 \in S, \ \alpha \in \mathbb{R})$

① $\mathcal{L}(f_1 + f_2) = \mathcal{L}(f_1) + \mathcal{L}(f_2)$

② $\mathcal{L}(\alpha f_1) = \alpha \mathcal{L}(f_1)$

i.e.

$\mathcal{L}$ is linear if $\forall\, s_1, s_2 \in S = \text{Funct}(\mathbb{Z} \to \mathbb{R})$

$\forall\, \alpha, \beta \in \mathbb{R}$ we have

$$\mathcal{L}(\alpha s_1 + \beta s_2) = \alpha \mathcal{L}(s_1) + \beta \mathcal{L}(s_2)$$

E.g. "$1 + 2^2 + \dots + n^2$" $= \dfrac{n(n+1)(2n+1)}{6} = p_2(n)$

$(\mathcal{D} f)(n) := f(n+1) - f(n)$ (any function)

$(\mathcal{D} p_2)(n) = (n+1)^2$

$\mathcal{D} : \text{Funct}(\mathbb{Z} \to \mathbb{R}) \longrightarrow \text{Funct}(\mathbb{Z} \to \mathbb{R})$,

$\mathcal{D}$ is linear:

$$\mathcal{D}(f_1 - f_2) = \mathcal{D} f_1 - \mathcal{D} f_2$$

$$\mathcal{D}(\alpha f_1) = \alpha \mathcal{D}(f_1)$$

Solve $(\mathcal{D} p) =$ "$(n+1)^2$"

Ah-ha  here's a solution $p(n) = \dfrac{n(n+1)(2n+1)}{6} + 2019$

If $(\mathcal{D}p) = (n+1)^2$ for some p

and $(\mathcal{D}q) = (n+1)^2 = (\mathcal{D}p)$

then

$$p - q = \ker(\mathcal{D}) = \{ \underset{f : \mathbb{Z} \to \mathbb{R}}{f \in \mathrm{Funct}(\mathbb{Z} \to \mathbb{R})} \mid (\mathcal{D}f) = 0_{\text{funct}} \}$$

$$= \{ f : \mathbb{Z} \to \mathbb{R} \mid f(n+1) - f(n) = 0 \ \forall n \in \mathbb{Z} \}$$

$$= \{ f : \mathbb{Z} \to \mathbb{R} \mid f = C \text{ is constant} \}$$

So

$$(\mathcal{D}q) = (n+1)^2 \implies q = p + C, \quad C \text{ const}$$

$$= \left(\frac{n(n+1)(2n+1)}{6} + 2019 \right) + C$$

If $q(1) = 1^2$ or $q(2) = 1^2 + 2^2$ or $q(0) = 0$

then

$$q(n) = \left(\frac{n(n+1)(2n+1)}{6} + 2019 \right) - C, \quad C = 2019$$

Example 2: (HW, Section 4)

Fibonacci numbers:

$\ldots, -8, 5, -3, 2, -1, 1, 0, \underset{F(1)}{\underset{\parallel}{1}}, \underset{F(2)}{\underset{\parallel}{1}}, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$

$n \geq 3 \qquad F(n) := F(n-1) + F(n-2) \quad : \quad$ $F(3) = 1+1 = 2$, $F(4) = 2+1 = 3$, $\vdots$

$\curvearrowright \quad F(n-2) := F(n) - F(n-1)$

$n \leq 2$

$F(0) = F(2) - F(1) = 0$

$F(-1) = F(1) - F(0) = 1$

$F(-2) = \ldots \qquad = -1$

=

Look at $\mathcal{L}_{Fib} : \text{Functions}(\mathbb{Z} \to \mathbb{R}) \longrightarrow \text{Function}(\mathbb{Z} \to \mathbb{R})$

$$\boxed{(\mathcal{L}_{Fib} f)(n) := f(n+2) - f(n+1) - f(n)}$$ is linear

=

What does $\ker(\mathcal{L}_{Fib}) = \{ \ldots ? \}$

$\text{Fibonacci}(n) \in \ker(\mathcal{L}_{Fib})$

$F: \quad \ldots, 0, 1, 1, 2, 3, 5, 8, 13, \ldots$

Fact: $\mathrm{Fib}(n) = \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right) \frac{1}{\sqrt{5}}$

Why?

When is

$$\begin{array}{ccccccccc} (\dots, & x^{-1}, & 1, & x, & x^2, & \dots) \in \ker(\mathcal{L}_{Fib}) \\ & & \| & \| & \| & \\ & f(-1) & f(0), & f(1), & f(2) & \end{array}$$

i.e. take any $x \in \mathbb{R}$, set $f(n) := x^n$

for which (if any) $x \in \mathbb{R}$ is $f \in \ker(\mathcal{L}_{Fib})$

iff

$$f(n+2) - f(n+1) - f(n) = 0 \quad \forall n \in \mathbb{Z}$$

$$x^{n+2} - x^{n+1} - x^n = 0 \quad \forall n \in \mathbb{Z}$$

$$x^2 - x - 1 = 0$$

$$x = \frac{1 \pm \sqrt{5}}{2} \quad \text{(just guess)}$$